

## Questions

**Example** Find the area of the region that is bounded by the given curve and lies in the given sector.

$$r = \sin \theta, \quad \pi/3 \leq \theta \leq 2\pi/3.$$

**Example** Sketch the curve and find the area it encloses.

$$r^2 = 4 \cos 2\theta.$$

**Example** Find the length of the polar curve

$$r = e^{2\theta}, \quad 0 \leq \theta \leq 2\pi.$$

## Solutions

**Example** Find the area of the region that is bounded by the given curve and lies in the given sector.

$$r = \sin \theta, \quad \pi/3 \leq \theta \leq 2\pi/3.$$

The area is given by the integral  $\frac{1}{2} \int_a^b f(\theta)^2 d\theta$ .

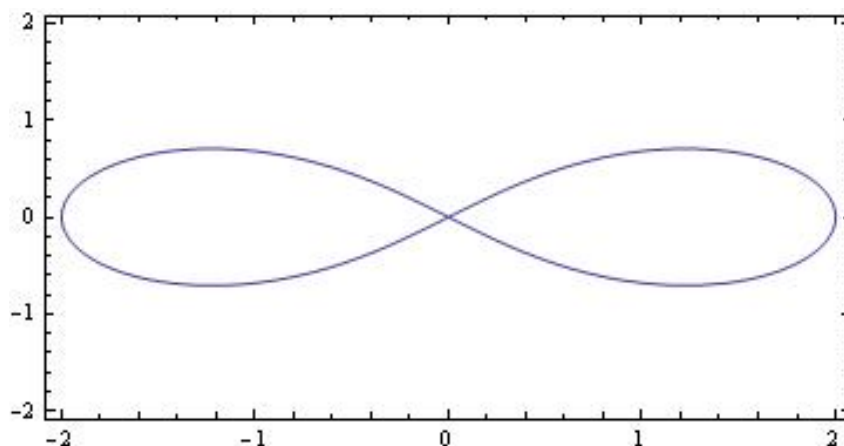
$$\begin{aligned} A &= \frac{1}{2} \int_a^b f(\theta)^2 d\theta \\ &= \frac{1}{2} \int_{\pi/3}^{2\pi/3} \sin^2 \theta d\theta \\ &= \frac{1}{2} \int_{\pi/3}^{2\pi/3} \frac{1}{2} (1 - \cos 2\theta) d\theta \\ &= \frac{1}{4} \left( \int_{\pi/3}^{2\pi/3} d\theta - \int_{\pi/3}^{2\pi/3} \cos 2\theta d\theta \right) \\ &= \frac{1}{4} \left( \theta \Big|_{\pi/3}^{2\pi/3} - \frac{1}{2} \sin 2\theta \Big|_{\pi/3}^{2\pi/3} \right) \\ &= \frac{1}{4} \left( \frac{2\pi}{3} - \frac{\pi}{3} - \frac{1}{2} \sin \frac{4\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} \right) \\ &= \frac{1}{4} \left( \frac{\pi}{3} - \frac{1}{2} \cdot \frac{-\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \right) \\ &= \frac{1}{4} \left( \frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) \end{aligned}$$

**Example** Sketch the curve and find the area it encloses.

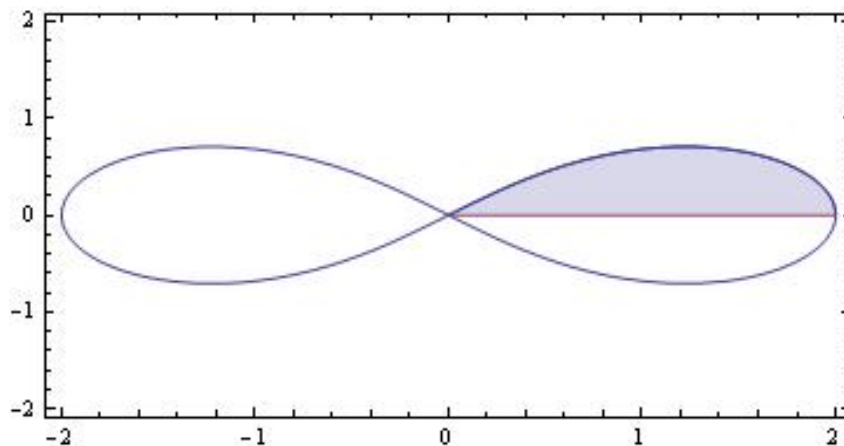
$$r^2 = 4 \cos 2\theta.$$

$$\begin{aligned} r^2 &= 4 \cos 2\theta \\ (x^2 + y^2) &= 4(\cos^2 \theta - \sin^2 \theta) \\ (x^2 + y^2) &= 4 \left( \frac{x^2}{r^2} - \frac{y^2}{r^2} \right) \\ (x^2 + y^2)^2 &= 4x^2 - 4y^2 \end{aligned}$$

`ContourPlot[(x^2 + y^2)^2 == 4x^2 - 4y^2, {x, -2, 2}, {y, -2, 2}, AxesLabel -> {"x", "y"}]`



The area is given by the integral  $\frac{1}{2} \int_a^b f(\theta)^2 d\theta$ . Since the function is periodic (cosine), we know it will repeat. The period of  $\cos x$  is  $2\pi$ , so the period of  $\cos 2x$  is  $\pi$ . From our sketch, if we go from  $0$  to  $\pi/4$  we will get one quarter the curve. The area swept out if  $0 \leq \theta \leq \pi/4$  is the shaded region below:



Therefore,

$$\begin{aligned}
 A &= \frac{1}{2} \int_a^b f(\theta)^2 d\theta \\
 \frac{1}{4}A &= \frac{1}{2} \int_0^{\pi/4} (\sqrt{4 \cos 2\theta})^2 d\theta \\
 &= \frac{1}{2} \int_0^{\pi/4} 4 \cos 2\theta d\theta \\
 &= 2 \int_0^{\pi/4} \cos 2\theta d\theta \\
 &= \frac{2}{2} \sin 2\theta \Big|_0^{\pi/4} \\
 &= \sin \frac{\pi}{2} - \sin 0 \\
 &= 1 \\
 A &= 4
 \end{aligned}$$

**Example** Find the length of the polar curve

$$r = e^{2\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Switch to parametric representation:

$$x = r \cos \theta = e^{2\theta} \cos \theta, \quad y = r \sin \theta = e^{2\theta} \sin \theta.$$

$$\frac{dx}{d\theta} = 2e^{2\theta} \cos \theta - e^{2\theta} \sin \theta$$

$$\frac{dy}{d\theta} = 2e^{2\theta} \sin \theta + e^{2\theta} \cos \theta$$

$$\begin{aligned}
 \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= (2e^{2\theta} \cos \theta - e^{2\theta} \sin \theta)^2 + (2e^{2\theta} \sin \theta + e^{2\theta} \cos \theta)^2 \\
 &= 4e^{4\theta} \cos^2 \theta + e^{4\theta} \sin^2 \theta + 4e^{4\theta} \sin^2 \theta + e^{4\theta} \cos^2 \theta \\
 &= 4e^{4\theta} + e^{4\theta} \\
 &= 5e^{4\theta} \\
 ds &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\
 &= \sqrt{5}e^{2\theta} d\theta \\
 L &= \int ds
 \end{aligned}$$

$$\begin{aligned} &= \int_0^{2\pi} \sqrt{5}e^{2\theta} d\theta \\ &= \frac{\sqrt{5}}{2} e^{2\theta} \Big|_0^{2\pi} \\ &= \frac{\sqrt{5}}{2} (e^{4\pi} - 1) \end{aligned}$$