Example (10.5.3) Find the area of the region that is bounded by the given curve and lies in the given sector.

$$r = \sin\theta, \quad \pi/3 \le \theta \le 2\pi/3.$$

The area is given by the integral $\frac{1}{2}\int_a^b f(\theta)^2\,d\theta$.

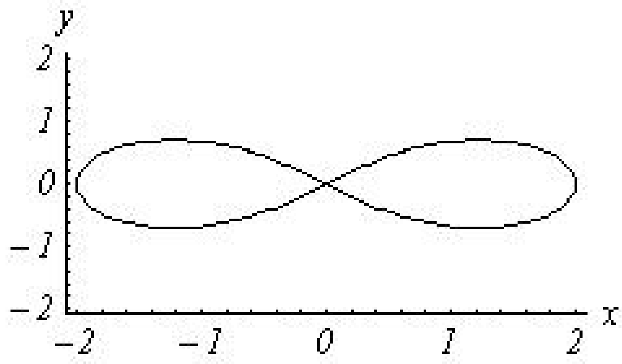
$$\begin{aligned}
A &= \frac{1}{2}\int_a^b f(\theta)^2\,d\theta \\
&= \frac{1}{2}\int_{\pi/3}^{2\pi/3} \sin^2\theta\,d\theta \\
&= \frac{1}{2}\int_{\pi/3}^{2\pi/3} \frac{1}{2}(1-\cos 2\theta)\,d\theta \\
&= \frac{1}{4}\left(\int_{\pi/3}^{2\pi/3} d\theta - \int_{\pi/3}^{2\pi/3}\cos 2\theta\,d\theta\right) \\
&= \frac{1}{4}\left(\theta|_{\pi/3}^{2\pi/3} - \frac{1}{2}\sin 2\theta\|_{\pi/3}^{2\pi/3}\right) \\
&= \frac{1}{4}\left(\frac{2\pi}{3} - \frac{\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right) \\
&= \frac{1}{4}\left(\frac{\pi}{3} - \frac{1}{2}\cdot\frac{-\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) \\
&= \frac{1}{4}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)
\end{aligned}$$

Example (10.5.11) Sketch the curve and find the area it encloses.

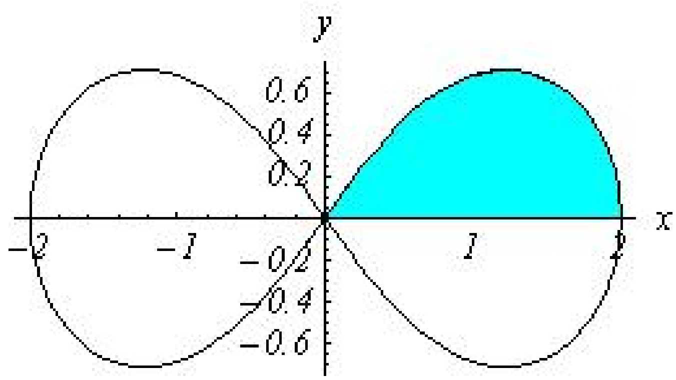
$$r^2 = 4\cos 2\theta.$$

$$\begin{aligned}
r^2 &= 4\cos 2\theta \\
(x^2+y^2) &= 4(\cos^2\theta - \sin^2\theta) \\
(x^2+y^2) &= 4\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right) \\
(x^2+y^2)^2 &= 4x^2 - 4y^2
\end{aligned}$$

```
<< Graphics`ImplicitPlot`
ImplicitPlot[(x^2 + y^2)^2 == 4x^2 - 4y^2, {x, -2, 2}, {y, -2, 2},
  AxesLabel -> {"x", "y"}]
```



The area is given by the integral $\frac{1}{2}\int_a^b f(\theta)^2\,d\theta$. Since the function is periodic (cosine), we know it will repeat. The period of $\cos x$ is 2π, so the period of $\cos 2x$ is π. From our sketch, if we go from 0 to $\pi/4$ we will get one quarter the curve. The area swept out if $0 \leq \theta\pi/4$ is the shaded region below:



Therefore,

$$
\begin{aligned}
A &= \frac{1}{2}\int_a^b f(\theta)^2\,d\theta \\
\frac{1}{4}A &= \frac{1}{2}\int_0^{\pi/4} (\sqrt{4\cos 2\theta})^2\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/4} 4\cos 2\theta\,d\theta \\
&= 2\int_0^{\pi/4} \cos 2\theta\,d\theta \\
&= \frac{2}{2}\sin 2\theta|_0^{\pi/4} \\
&= \sin\frac{\pi}{2} - \sin 0 \\
&= 1 \\
A &= 4
\end{aligned}
$$

Example (10.5.3) Find the length of the polar curve

$$r = e^{2\theta}, \quad 0 \le \theta \le 2\pi.$$

Switch to parametric representation:

$$x = r\cos\theta = e^{2\theta}\cos\theta, \quad y = r\sin\theta = e^{2\theta}\sin\theta.$$

$$\frac{dx}{d\theta} = 2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta$$

$$\frac{dy}{d\theta} = 2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta$$

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(2e^{2\theta}\cos\theta - e^{2\theta}\sin\theta\right)^2 + \left(2e^{2\theta}\sin\theta + e^{2\theta}\cos\theta\right)^2 \\
&= 4e^{4\theta}\cos^2\theta + e^{4\theta}\sin^2\theta + 4e^{4\theta}\sin^2\theta + e^{4\theta}\cos^2\theta \\
&= 4e^{4\theta} + e^{4\theta} \\
&= 5e^{4\theta} \\
ds &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \sqrt{5}e^{2\theta}\, d\theta \\
L &= \int ds \\
&= \int_0^{2\pi} \sqrt{5}e^{2\theta}\, d\theta \\
&= \frac{\sqrt{5}}{2}e^{2\theta}\Big|_0^{2\pi} \\
&= \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)
\end{aligned}$$