Questions

1) Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \dfrac{a_n}{2} & a_n \text{ even} \\ 3a_n + 1 & a_n \text{ odd} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

2) For what values of r is the sequence $\{nr^n\}$ convergent?

3) Find the limit of the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$.

4) A sequence is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.

(a) By induction or otherwise, show $\{a_n\}$ is increasing and bounded above by 3. Show the sequence is convergent.

(b) Find $\lim_{n\to\infty} a_n$.

Solutions

1) We could work this out by hand, but let's extend our knowledge of *Mathematica* a little instead.

New commands are `If, OddQ`. There is also a command `EvenQ`, but we won't need it for this problem. The original sequence was given as

$$a_1 = 11, \quad a_{n+1} = \begin{cases} \frac{a_n}{2} & a_n \text{ even} \\ 3a_n + 1 & a_n \text{ odd} \end{cases}$$

which has $n = 1, 2, 3 \ldots$. However, to input it into *Mathematica* we prefer the following

$$a_1 = 11, \quad a_n = \begin{cases} \frac{a_{n-1}}{2} & a_{n-1} \text{ even} \\ 3a_{n-1} + 1 & a_{n-1} \text{ odd} \end{cases}$$

which has $n = 2, 3, 4 \ldots$.

Here are the *Mathematica* commands to define the sequence:

```
a[1] := 11
a[n_] := a[n] = If[OddQ[a[n - 1]], 3a[n - 1] + 1, a[n - 1]/2]
```

I treated the sequence with different starting value as a totally new sequence, and defined it as

```
b[1] := 25
b[n_] := b[n] = If[OddQ[b[n - 1]], 3b[n - 1] + 1, b[n - 1]/2]
```

The sequences are found to be:

$\{a_n\} = \{11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4\}$

$\{b_n\} = \{25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2,$
$1, 4, 2, 1, 4\}$

To make the connection easier to see, let's write a small table.

n	a_n	b_n
18	1	10
19	4	5
20	2	16
21	1	8
22	4	4
23	2	2
24	1	1
25	4	4
26	2	2

The two sequences become the same after the $n = 22$! This isn't surprising, since the only thing that changed was the initial starting point...but then again, it is surprising, since a different starting point you might think would lead to different values later on. The sequence is oscillating, so it does not converge.

There are a number of conjectures you might make based on your findings. My conjecture is that for n sufficiently large, all sequences of this form, regardless of the value of a_1, will look like $\{..., 4, 2, 1, 4, 2, 1, ...\}$. This is just a conjecture, to prove this would require more work!

2) This is very similar to the sequence $\{r^n\}$, which is discussed in the text. We are assuming that $n = 1, 2, 3, \ldots$.

To begin, the sequence $\{nr^n\}$ will be divergent if $r > 1$, since the sequence will undergo exponential growth.

If $r = 1$, the sequence will become $\{n\}$, which exhibits linear growth, and so the sequence will diverge.

If $r < -1$, the sequence will oscillate with exponential growth, and so the sequence will diverge.

If $r = -1$, the sequence will become $\{n(-1)^n\}$, the sequence will oscillate with linear growth, and so the sequence will diverge.

If $r = 0$, the sequence will become $\{0\} = \{0, 0, 0, 0, \ldots\}$ which is convergent to zero.

All that remains is what happens if $-1 < r < 1$, $r \neq 0$. So let's assume this is the case and proceed.

We will need to work out a limit that is easiest to deal with in the continuous form. Let $f(x) = xr^x$, and we have $f(n) = nr^n$, $n = 1, 2, 3, \ldots$.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} xr^x \longrightarrow \infty \cdot 0 \\
&= \lim_{x\to\infty} \frac{x}{r^{-x}} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminant quotient, l'Hospital's rule applies} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}r^{-x}}
\end{aligned}$$

$$\begin{aligned} &= -\lim_{x\to\infty} \frac{1}{r^{-x}} \\ &= -\lim_{x\to\infty} r^x \\ &= 0 \end{aligned}$$

The above analysis depended heavily on the fact that $-1 < r < 1$, $r \neq 0$. The limits were all taken with this restriction on r in place.

Since $\lim_{x\to\infty} f(x) = 0$ and $f(n) = nr^n$, $n = 1, 2, 3, \ldots$, we can say that $\lim_{n\to\infty} nr^n = 0$ if $-1 < r < 1$, $r \neq 0$.

We have now treated all possible values of r. We see that the sequence $\{nr^n\}$ converges to zero if $-1 < r < 1$, and diverges for all other values of r.

3) (Note: alternate solution follows this solution.)

We can write the sequence we are investigating recursively as follows

$$a_1 = \sqrt{2}, \quad a_n = \sqrt{2a_{n-1}}, \quad n = 2, 3, 4, \ldots$$

Show the sequence is increasing

To show the sequence is increasing we shall use mathematical induction (see page 79).

We want to show the result $a_{n+1} \geq a_n$ all $n \geq 1$.

- Step 1 in induction: Show the result is true for $n = 1$.
  If $n = 1$, we have that
  $$a_2 = \sqrt{2\sqrt{2}} = 2^{3/4} = \left(2^{1/4}\right)^3 > \left(2^{1/4}\right)^2 = \sqrt{2} = a_1$$
  So the result is true for $n = 1$.
- Step 2 in induction: Assume the result is true for $n = k$.
  We assume $a_{k+1} \geq a_k$.
- Step 3 in induction: Show the result is true for $n = k + 1$.
  From Step 2, we have:
  $$\begin{aligned} a_{k+1} &\geq a_k \\ 2a_{k+1} &\geq 2a_k \\ \sqrt{2a_{k+1}} &\geq \sqrt{2a_k} \\ a_{k+2} &\geq a_{k+1} \end{aligned}$$
  So we have shown that $a_{k+2} \geq a_{k+1}$ true.

Therefore, $a_{n+1} \geq a_n$ for all $n \geq 1$ is true by mathematical induction.

The sequence is increasing. The lower bound of the sequence is $a_1 = \sqrt{2}$.

Show the sequence is bounded above

To show the sequence is bounded above we shall again use mathematical induction.

We want to show the result $a_n < 5$ all $n \geq 1$. I picked 5 out of the air. If it doesn't work, I will try something else.

- Step 1 in induction: Show the result is true for $n = 1$.
  If $n = 1$, we have that

  $$a_1 = \sqrt{2} < 5$$

  So the result is true for $n = 1$.
- Step 2 in induction: Assume the result is true for $n = k$.
  We assume $a_k \leq 5$.
- Step 3 in induction: Show the result is true for $n = k + 1$.
  From Step 2, we have:

  $$\begin{aligned} a_k &\leq 5 \\ 2a_k &\leq 2(5) \\ a_{k+1} = \sqrt{2a_k} &\leq \sqrt{10} < 5 \\ a_{k+1} &\leq 5 \end{aligned}$$

  So we have shown that $a_{k+1} \leq 5$ true.

Therefore, $a_n \leq 5$ for all $n \geq 1$ is true by mathematical induction.

The sequence is bounded above by 5.

Find the limit of the sequence

Since the sequence is increasing, it is monotonic. The sequence is also bounded. Any monotonic, bounded sequence is convergent (by the monotonic sequence theory). Therefore, the sequence is convergent.

Since the sequence converges, it must be true that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = L$$

We can therefore say

$$\begin{aligned} a_n &= \sqrt{2a_{n-1}} \\ \lim_{n\to\infty}(a_n &= \sqrt{2a_{n-1}}) \\ \lim_{n\to\infty} a_n &= \sqrt{2\lim_{n\to\infty} a_{n-1}} \\ L &= \sqrt{2L} \\ L^2 &= 2L \\ L &= 0, +2 \end{aligned}$$

We can exclude $L = 0$, since the sequence is increasing, and $a_1 = \sqrt{2} > 0$. Therefore, the limit of the sequence is 2.

Alternate solution to Problem 3

Rather than looking for a recursive definition of the sequence, we could instead search for the general term a_n. In this case, we can find it.

$$\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$$
$$\{2^{1/2}, (2^{3/2})^{1/2}, (2 \cdot 2^{3/4})^{1/2}, \ldots\}$$
$$\{2^{1/2}, 2^{3/4}, (2^{7/4})^{1/2}, \ldots\}$$
$$\{2^{1/2}, 2^{3/4}, 2^{7/8}, \ldots\}$$

The exponent is given by $(2^n - 1)/2^n$. This sequence can be expressed as $\{a_n\}_{n=1}^{\infty} = \{2^{(2^n-1)/2^n}\}$.

Now that we have this, we don't need to use induction. We can work directly with the term a_n to calculate the limit.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} 2^{(2^n-1)/2^n} \\
&= 2^{\lim_{n\to\infty}(2^n-1)/2^n} \\
\lim_{n\to\infty} \frac{(2^n-1)}{2^n} &= \lim_{n\to\infty} (1 - 2^{-n}) \\
&= 1 \\
\lim_{n\to\infty} a_n &= 2^{\lim_{n\to\infty}(2^n-1)/2^n} = 2^1 = 2
\end{aligned}$$

4) We can write the sequence we are investigating recursively as follows

$$a_1 = \sqrt{2}, \quad a_n = \sqrt{2 + a_{n-1}}, \quad n = 2, 3, 4, \ldots$$

Show the sequence is increasing

To show the sequence is increasing we shall use mathematical induction.

We want to show the result $a_{n+1} \geq a_n$ all $n \geq 1$.

- Step 1 in induction: Show the result is true for $n = 1$.
  If $n = 1$, we have that
  $$a_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2 + 1} = \sqrt{3} > \sqrt{2} = a_1$$
  So the result is true for $n = 1$.
- Step 2 in induction: Assume the result is true for $n = k$.
  We assume $a_{k+1} \geq a_k$.

- Step 3 in induction: Show the result is true for $n = k + 1$.
  From Step 2, we have:

$$\begin{aligned} a_{k+1} &\geq a_k \\ 2 + a_{k+1} &\geq 2 + a_k \\ \sqrt{2 + a_{k+1}} &\geq \sqrt{2 + a_k} \\ a_{k+2} &\geq a_{k+1} \end{aligned}$$

  So we have shown that $a_{k+2} \geq a_{k+1}$ true.

Therefore, $a_{n+1} \geq a_n$ for all $n \geq 1$ is true by mathematical induction.

The sequence is increasing. The lower bound of the sequence is $a_1 = \sqrt{2}$.

Show the sequence is bounded above

To show the sequence is bounded above we shall again use mathematical induction.

We want to show the result $a_n < 3$ all $n \geq 1$. The 3 was given to us in the problem.

- Step 1 in induction: Show the result is true for $n = 1$.
  If $n = 1$, we have that

$$a_1 = \sqrt{2} < 3$$

  So the result is true for $n = 1$.
- Step 2 in induction: Assume the result is true for $n = k$.
  We assume $a_k \leq 3$.
- Step 3 in induction: Show the result is true for $n = k + 1$.
  From Step 2, we have:

$$\begin{aligned} a_k &\leq 3 \\ 2 + a_k &\leq 2 + 3 \\ a_{k+1} = \sqrt{2 + a_k} &\leq \sqrt{5} < 3 \\ a_{k+1} &\leq 3 \end{aligned}$$

  So we have shown that $a_{k+1} \leq 3$ true.

Therefore, $a_n \leq 5$ for all $n \geq 1$ is true by mathematical induction.

The sequence is bounded above by 3.

Find the limit of the sequence

Since the sequence is increasing, it is monotonic. The sequence is also bounded. Any monotonic, bounded sequence is convergent (by the monotonic sequence theory). Therefore, the sequence is convergent.

Since the sequence converges, it must be true that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = L$$

We can therefore say

$$\begin{aligned}
a_n &= \sqrt{2+a_{n-1}} \\
\lim_{n\to\infty}(a_n &= \sqrt{2+a_{n-1}}) \\
\lim_{n\to\infty} a_n &= \sqrt{2+\lim_{n\to\infty} a_{n-1}} \\
L &= \sqrt{2+L} \\
L^2 &= 2+L \\
L^2-L-2 &= 0 \\
L &= -1, +2
\end{aligned}$$

We can exclude $L=-1$, since the sequence is increasing, and $a_1 = \sqrt{2} > 0$. We could also justify excluding -1 since $a_n = +\sqrt{2+a_{n-1}} > 0$.

Therefore, the limit of the sequence is 2.