Questions

1) Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

2) Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

$$\sum_{n=4}^{\infty} \frac{3}{n(n-1)}$$

3) If the nth partial sum of a series $\displaystyle\sum_{n=1}^{\infty} a_n$ is $s_n = \dfrac{n-1}{n+1}$, find a_n and $\displaystyle\sum_{n=1}^{\infty} a_n$.

4) Graph the curves $y = x^n$, $0 \le x \le 1$, for $n = 0, 1, 2, 3, 4, \ldots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Solutions

1) In this problem, we are given a series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$.

The series is built from a sequence $\{a_n\} = \{n/(n+1)\}$ from which a sequence of partial sums $\{s_n\} = \{\sum_{i=1}^{n} a_i\} = \{\sum_{i=1}^{n} i/(i+1)\}$ is formed.

If the series converges, then the sequence of partial sums must converge. This is what we are investigating in this problem.

Here are the *Mathematica* commands I used to create the plots. You could also do this by hand if you like. I've added in some `PlotStyle` commands to make the plots clearer.

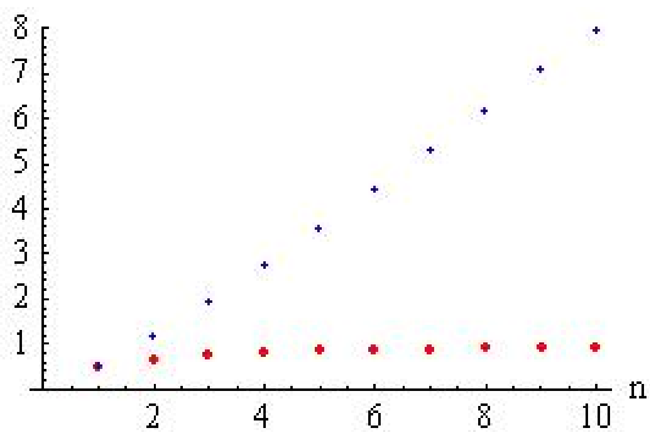
```
Clear[a, s]
a[i_] = i/(i + 1);
seq = Table[{i, a[i]}, {i, 1, 10}]
plotseq = ListPlot[seq, PlotStyle -> {RGBColor[1, 0, 0], PointSize[0.02]}]

s[n_] = Sum[a[i], {i, 1, n}];
seqpartialsums = Table[{n, s[n]}, {n, 1, 10}]
```

```
plotpartialsumseq = ListPlot[seqpartialsums, PlotStyle -> {RGBColor[0, 0, 1], PointSize[0.02]}]
Show[plotseq, plotpartialsumseq, AxesLabel -> {"n", ""},
  PlotRange -> {Automatic, {0, 8}}]
```

From the above output, I can easily read off the first ten terms in the sequence and the sequence of partial sums. They are

$$\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{10}{11}, \ldots\right\}$$

$$\{s_n\} = \left\{\frac{1}{2}, \frac{7}{6}, \frac{23}{12}, \frac{163}{60}, \frac{71}{20}, \frac{617}{140}, \frac{1479}{280}, \frac{15551}{2520}, \frac{17819}{2520}, \frac{221209}{27720}, \ldots\right\}$$



The large red dots are the sequence of terms a_n, and the smaller blue dots are the sequence of partial sums, s_n.

It appears that the series is divergent, since it appears that the sequence of partial sums is divergent,

$$\lim_{n\to\infty} s_n = \infty.$$

We can prove the series is divergent by showing $\lim_{n\to\infty} a_n \neq 0$.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{n+1} \\
&= \lim_{n\to\infty} \frac{1}{1+1/n} \\
&= \frac{1}{1+0} = 1 \neq 0
\end{aligned}$$

Since $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ is divergent by *the test for divergence.*

2) In this problem, we are given a series $\displaystyle\sum_{n=4}^{\infty} \frac{3}{n(n-1)}$.

The series is built from a sequence $\{a_n\}_{n=4}^{\infty} = \{3/(3(n-1))\}_{n=4}^{\infty}$ from which a sequence of partial sums $\{s_n\}_{n=4}^{\infty} = \{\sum_{i=4}^{n} a_i\}_{n=4}^{\infty} = \{\sum_{i=4}^{n} 3/(i(i-1))\}_{n=4}^{\infty}$ is formed.

If the series converges, then the sequence of partial sums must converge. This is what we are investigating in this problem.

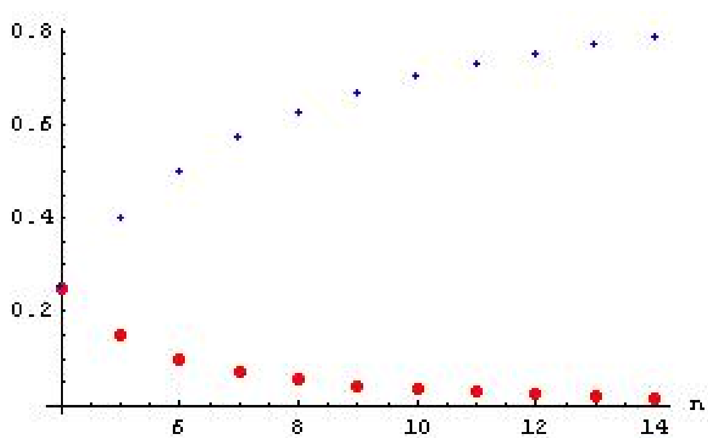
Here are the *Mathematica* commands I used to create the plots. You could also do this by hand if you like. I've added in some `PlotStyle` commands to make the plots clearer.

```
Clear[a, s]
a[i_] = 3/i/(i - 1);
seq = Table[{i, a[i]}, {i, 4, 14}]
plotseq = ListPlot[seq, PlotStyle -> {RGBColor[1, 0, 0], PointSize[0.02]}]

s[n_] = Sum[a[i], {i, 4, n}];
seqpartialsums = Table[{n, s[n]}, {n, 4, 14}]
plotpartialsumseq =
  ListPlot[seqpartialsums, PlotStyle -> {RGBColor[0, 0, 1], PointSize[0.015]}]
Show[plotseq, plotpartialsumseq, AxesLabel -> {"n", ""}]
```

From the above output, I can easily read off the first ten terms in the sequence and the sequence of partial sums. They are

$$\{a_n\}_{n=4}^{\infty} = \left\{\frac{1}{4}, \frac{3}{20}, \frac{1}{10}, \frac{1}{14}, \frac{3}{56}, \frac{1}{24}, \frac{1}{30}, \frac{3}{110}, \frac{1}{44}, \frac{1}{52}, \frac{3}{182}, \ldots\right\}$$

$$\{s_n\}_{n=4}^{\infty} = \left\{\frac{1}{4}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7}, \frac{5}{8}, \frac{2}{3}, \frac{7}{10}, \frac{8}{11}, \frac{3}{4}, \frac{10}{13}, \frac{11}{14}, \ldots\right\}$$



The large red dots are the sequence of terms a_n, and the smaller blue dots are the sequence of partial sums, s_n.

It appears that the series is convergent, since it appears that the sequence of partial sums is approaching 0.8.

To actually find the sum of the series, we need to be able to perform some manipulations on the partial sums to get rid of the summation, so we can take the limit.

The manipulation we are about to perform is *absolutely necessary*, since without it all we are able to write is

$$\sum_{n=4}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=4}^{n} \frac{3}{i(i-1)} = \sum_{i=4}^{\infty} \frac{3}{i(i-1)}$$

and we are not able to get the sum of the series.

The manipulation required here is partial fractions.

$$s_n = \sum_{i=4}^{n} \frac{3}{i(i-1)}$$

Partial fractions:

$$\begin{aligned} \frac{3}{i(i-1)} &= \frac{A}{i} + \frac{B}{i-1} \quad \text{split} \\ 3 &= A(i-1) + B(i) \quad \text{clear fractions} \end{aligned}$$

If we take $i = +1$, we get

$$3 = B(+1) \longrightarrow B = 3.$$

If we take $i = 0$, we get

$$3 = A(-1) \longrightarrow A = -3.$$

Therefore, we have

$$\frac{3}{i(i-1)} = \frac{-3}{i} + \frac{3}{i-1}$$

Now, we can simplify the partial sum

$$\begin{aligned} s_n &= \sum_{i=4}^{n} \frac{3}{i(i-1)} \\ &= \sum_{i=4}^{n} \left(-\frac{3}{i} + \frac{3}{i-1} \right) \\ &= -3\left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n-1} + \frac{1}{n} \right) + 3\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n-1} \right) \\ &= 3\left(\frac{1}{3} - \frac{1}{n} \right) \\ &= 1 - \frac{3}{n} \end{aligned}$$

Now, we can take the limit and get a meaningful answer:

$$\sum_{n=4}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{3}{n}\right) = 1.$$

We have shown that the series sums to 1. Pretty neat, eh?

3) We are given $s_n = \dfrac{n-1}{n+1}$.

The partial sums for this series are

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n, \quad n = 1, 2, 3, 4, \ldots,$$

and we also have

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1}, \quad n = 2, 3, 4, \ldots,$$

Subtracting, we find an expression for a_n in terms of the partial sums:

$$\begin{aligned} a_n &= s_n - s_{n-1}, \quad n = 2, 3, 4, \ldots, \\ &= \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} \\ &= \frac{n-1}{n+1} - \frac{n-2}{n} \\ &= \frac{n(n-1) - (n+1)(n-2)}{n(n+1)} \\ &= \frac{n^2 - n - (n^2 - n - 2)}{n(n+1)} \\ &= \frac{2}{n(n+1)} \end{aligned}$$

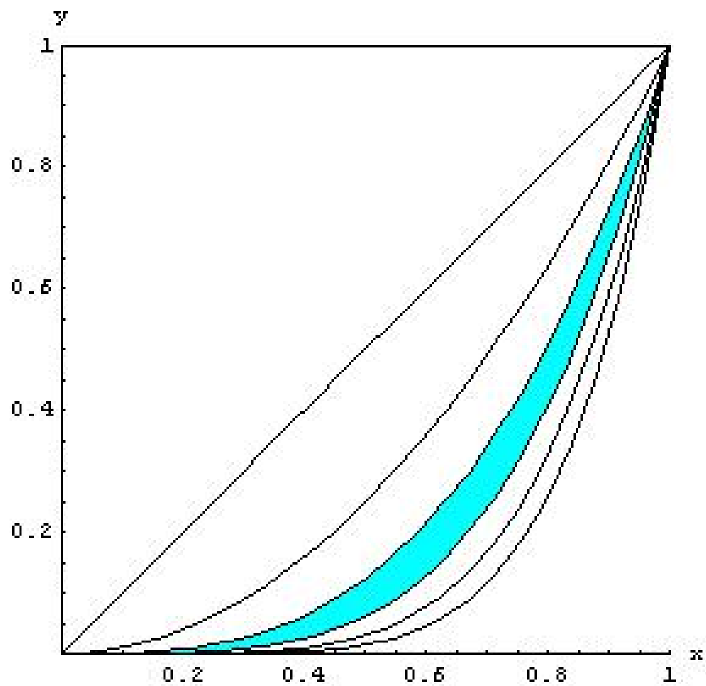
Since we used s_{n-1} to derive this result, it is only valid for $n = 2, 3, 4, 5, \ldots$.

To get a_1, we note that $a_1 = s_1$, so $a_1 = (1-1)/(1+1) = 0$.

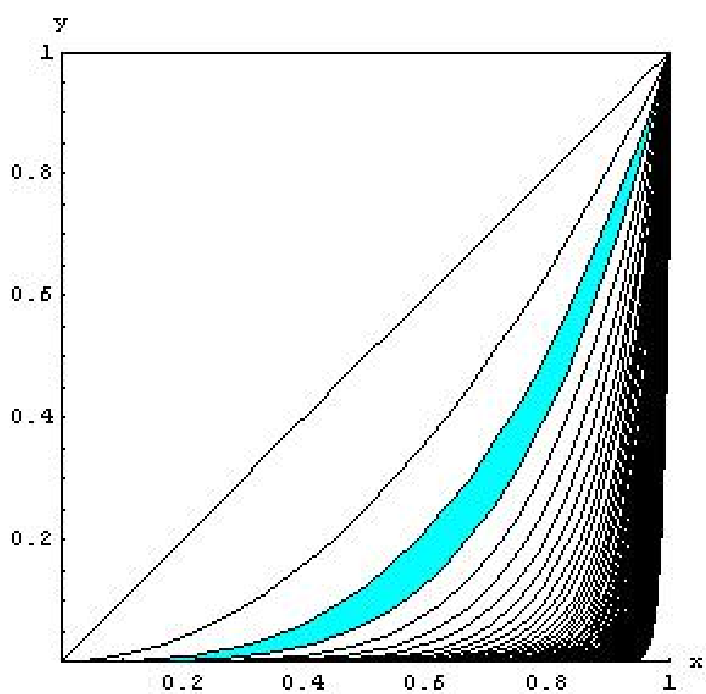
The sum of the series is given by

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right) = \lim_{n\to\infty} \left(\frac{1-1/n}{1+1/n}\right) = \left(\frac{1-0}{1+0}\right) = 1.$$

4) Here is a sketch of the functions $y = x^n$, $0 \le x \le 1$ for $n = 0, 1, 2, \ldots, 6$ (it is easy to draw by hand):



If we continue to sketch $y = x^n$, $n = 0, 1, 2, 3, 4, \ldots$, we will have curves which get closer and closer together and move toward the bottom and right hand side of the square of side 1.



The shaded region is the area between x^3 and x^4. This is given by an integral, $\int_0^1 (x^3 - x^4)\, dx$.

If we add up the area between all the curves, x^n and x^{n+1} for $n = 0, 1, 2, 3, \ldots$, we should get 1, since since this will be the area of the square with side 1.

$$\begin{aligned}
\sum_{n=0}^{\infty} \int_0^1 \left(x^n - x^{n+1}\right) dx &= 1 \\
\sum_{n=0}^{\infty} \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)_0^1 &= 1 \\
\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) &= 1 \\
\sum_{n=0}^{\infty} \frac{n+2-n-1}{(n+1)(n+2)} &= 1 \\
\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} &= 1
\end{aligned}$$

This is not quite what is asked for. However, we can simply relabel the dummy index $n = m - 1$, which means $m = n + 1$ and when $n = 0$ we have $m = 1$, and we get

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{m=1}^{\infty} \frac{1}{m(m+1)} = 1.$$