Example (11.9.4) Express $\dfrac{x}{1-x}$ as a power series, and find the interval of convergence.

The basic result we need is the geometric series, which is given as

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n, \text{ if } |y| < 1$$

Now, we manipulate the given function until we can utilize the geometric series result.

$$\begin{aligned} \frac{x}{1-x} &= x \cdot \left(\frac{1}{1-(x)} \right) \\ &= x \cdot \left(\sum_{n=0}^{\infty} (x)^n \right), \text{ if } |\,(x)\,| < 1 \\ &= \sum_{n=0}^{\infty} x^{n+1}, \text{ if } |x| < 1 \end{aligned}$$

Example (11.9.5) Express $\dfrac{1}{1-x^3}$ as a power series, and find the interval of convergence.

The basic result we need is the geometric series, which is given as

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n, \text{ if } |y| < 1$$

Now, we manipulate the given function until we can utilize the geometric series result.

$$\begin{aligned} \frac{1}{1-x^3} &= \frac{1}{1-(x^3)} \\ &= \sum_{n=0}^{\infty} \left(x^3\right)^n, \text{ if } |\left(x^3\right)| < 1 \\ &= \sum_{n=0}^{\infty} x^{3n}, \text{ if } |x| < 1 \end{aligned}$$

Example (11.9.17) Express $f(x) = \ln(5-x)$ as a power series, and find the interval of convergence.

The basic result we need is the geometric series, which is given as

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n, \text{ if } |y| < 1$$

The logarithm function doesn't look like $\frac{1}{1-y}$, but if we take a derivative we get:

$$\frac{d}{dx}\ln(5-x) = -\frac{1}{5-x} \longrightarrow \ln(5-x) = \int\left(-\frac{1}{5-x}\right)\,dx,$$

so we see that we can express the derivative of the logarithm as a power series, then we can integrate this power series term-by-term to get a power series for $f(x)$. Now that we know what to do, let's do it.

$$\begin{aligned}
-\frac{1}{5-x} &= -\frac{1}{5}\cdot\left(\frac{1}{1-(x/5)}\right) \\
&= -\frac{1}{5}\cdot\left(\sum_{n=0}^{\infty}(x/5)^n\right),\ \text{if } |\,(x/5)\,| < 1 \\
&= -\frac{1}{5}\cdot\left(\sum_{n=0}^{\infty}\frac{x^n}{5^n}\right),\ \text{if } |x| < 5 \\
&= -\left(\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}\right),\ \text{if } |x| < 5 \\
\ln(5-x) &= \int\left(-\frac{1}{5-x}\right)\,dx \\
&= \int\left(-\left(\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}\right)\right)\,dx,\ \text{if } |x| < 5 \\
&= -\int\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}\,dx,\ \text{if } |x| < 5 \\
&= -\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}\int x^n\,dx,\ \text{if } |x| < 5 \\
&= -\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}\cdot\frac{x^{n+1}}{n+1} + K,\ \text{if } |x| < 5 \\
&= -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}} + K,\ \text{if } |x| < 5
\end{aligned}$$

We still have to determine the constant of integration K. We want to choose a value of x to evaluate at that will *not* leave us with K in terms of an infinite series. All the terms in the infinite series will be zero if we choose $x = 0$. $x = 0$ is also within the interval of convergence for our series, so we know this is a valid choice.

This leads to

$$\begin{aligned}
\ln(5-x)\big|_{x=0} &= -\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)5^{n+1}}\Bigg|_{x=0} + K,\ \text{if } |x| < 5 \\
\ln(5-0) &= -\sum_{n=0}^{\infty}\frac{(0)^{n+1}}{(n+1)5^{n+1}} + K, \\
\ln 5 &= -(0+0+0+\cdots) + K, \\
\ln 5 &= K,
\end{aligned}$$

Now that we know the value of K, we are done, and the power series for $\ln(5-x)$ is given by

$$\ln(5-x) = \ln 5 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}, \text{ if } |x| < 5.$$

NOTE: This answer is the same as the answer given in the text, which is

$$\ln(5-x) = \ln 5 - \sum_{m=1}^{\infty} \frac{x^{m}}{m5^{m}}, \text{ if } |x| < 5.$$

Example (11.9.37) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation

$$f'(x) = f(x).$$

From this, deduce that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

To show the function is a solution of the differential equation, we have to show that it satisfies the differential equation. In this case, we have to show that the derivative of the series is equal to the original series.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d}{dx}x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}nx^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{n}{n!}x^{n-1} \\
&= 0 + 1 + \frac{2}{2!}x^1 + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \cdots \\
&= 1 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \\
&= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}x^n \\
&= f(x)
\end{aligned}$$

Therefore, $f(x)$ satisfies the differential equation $f'(x) = f(x)$.

Let's solve the differential equation using our regular separable technique. It is easier to use $y = f(x)$ when doing separation of variables.

$$\begin{aligned}
f'(x) &= f(x) \\
y' &= y \\
\frac{dy}{dx} &= y \\
\frac{dy}{y} &= dx \\
\int \frac{dy}{y} &= \int dx \\
\ln|y| &= x + c_1 \\
|y| &= e^{x+c_1} \\
y &= \pm e^{c_1} e^x \\
y &= Ae^x
\end{aligned}$$

where the constant of integration is $A = \pm e^{c_1}$.

Therefore, we have found the solution to the differential equation two ways. We can assume that the solution is unique, so the solutions must be the same. Therefore,

$$Ae^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots.$$

We can determine the constant A by evaluating at a convenient value of x. If we choose $x = 0$, then we get

$$Ae^0 = 1 + 0 + 0 + 0 + \cdots \longrightarrow A = 1,$$

and we have shown that

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$