Questions

Example Find the volume of the region which is created when we rotate the region below $y = 2x^2 - x^3$ and above $0 < x < 2$ about the y-axis.

Example Find the volume of the region which is created when we rotate the region below $y = \sqrt{x-1}$ and above $2 < x < 5$ about the x-axis.

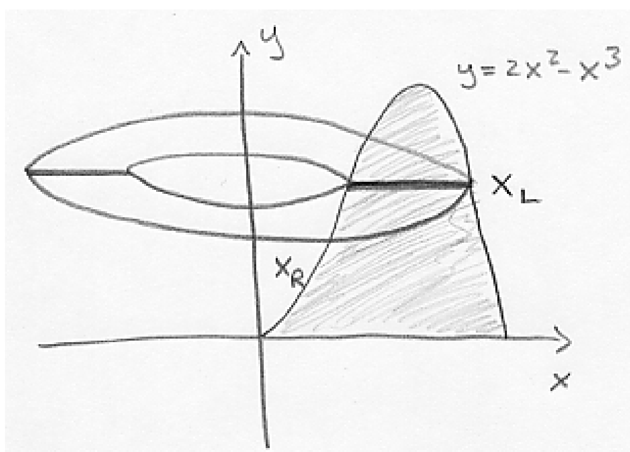
Example You have two spherical wooden balls of different radii, and want to make two beads. You drill a hole in one, and a hole in the other. For aesthetic reasons, the two holes have different radii. You discover that the height of both balls when you are done is the same. The mathematician inside you forces you to ask: "Which ball has more wood in it?"

Example Find the volume of the region which is created when we rotate the region bounded by the curves $y = 4x - x^2$ and $y = 8x - 2x^2$ about the axis $x = -2$.

Solutions

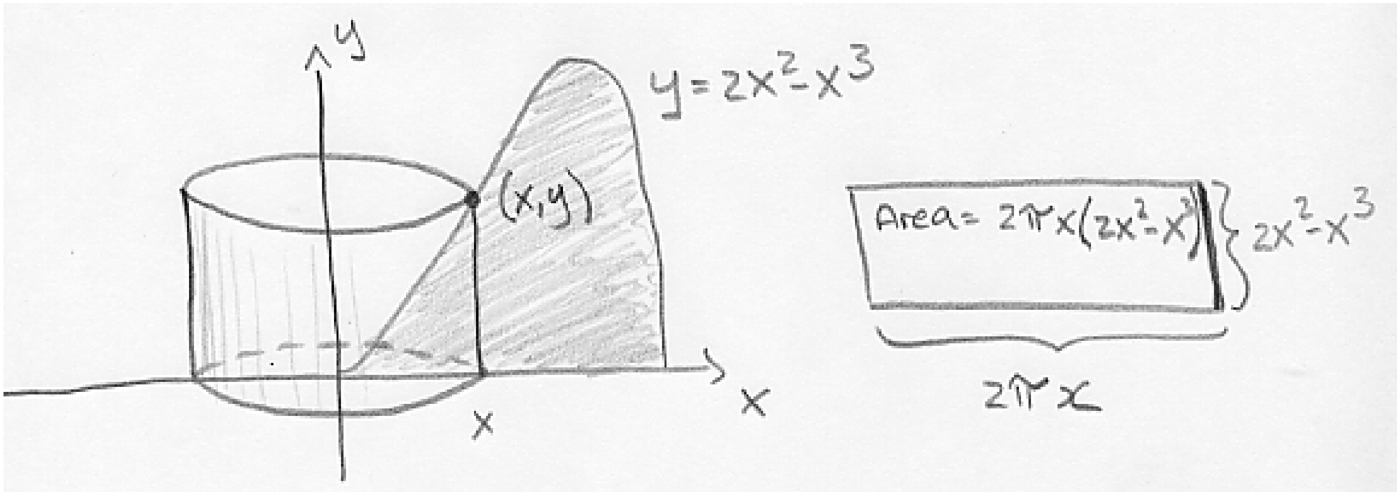
Example Find the volume of the region which is created when we rotate the region below $y = 2x^2 - x^3$ and above $0 < x < 2$ about the y-axis.

Solution First try to use the method of washers. Begin with a sketch:



We run into a problem determining x_R and x_L. We would need to solve $y = 2x^2 - x^3$ for x. *Mathematica* can do this for you (try it!), but you still probably don't want to work with the functions you find (messy!).

Instead, let's try the method of cylindrical shells. Begin with a sketch.

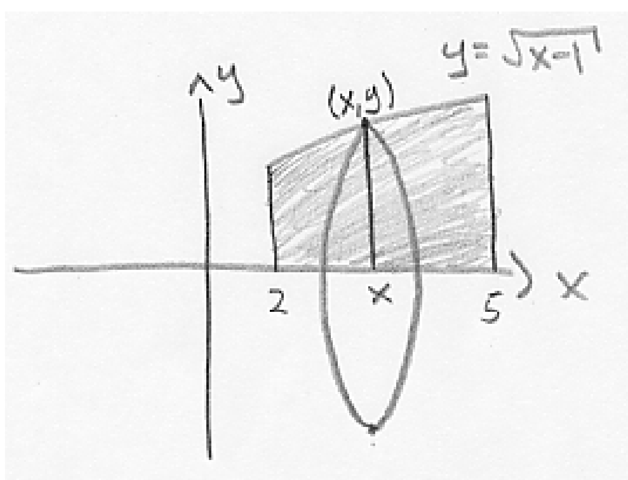


Integration limits: if $0 < x < 2$, the cylinder sweeps out the total volume. We integrate with respect to x.
Cylinder height $= y = 2x^2 - x^3$.
Cylinder radius $= x$.
Cylinder circumference $= 2\pi \times (\text{radius}) = 2\pi x$.
Cylinder surface area $= 2\pi x(2x^2 - x^3) = 2\pi(2x^3 - x^4)$.

$$\text{Volume} = \int_a^b (\text{area of cylinder})\, dx = \int_0^2 2\pi(2x^3 - x^4)\, dx = 2\pi \left(2\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_0^2 = 2\pi\left(\frac{2^4}{2} - \frac{2^5}{5}\right) = \frac{16\pi}{5}$$

Example Find the volume of the region which is created when we rotate the region below $y = \sqrt{x-1}$ and above $2 < x < 5$ about the x-axis.

Solution First, let's use washers. Begin with a sketch.

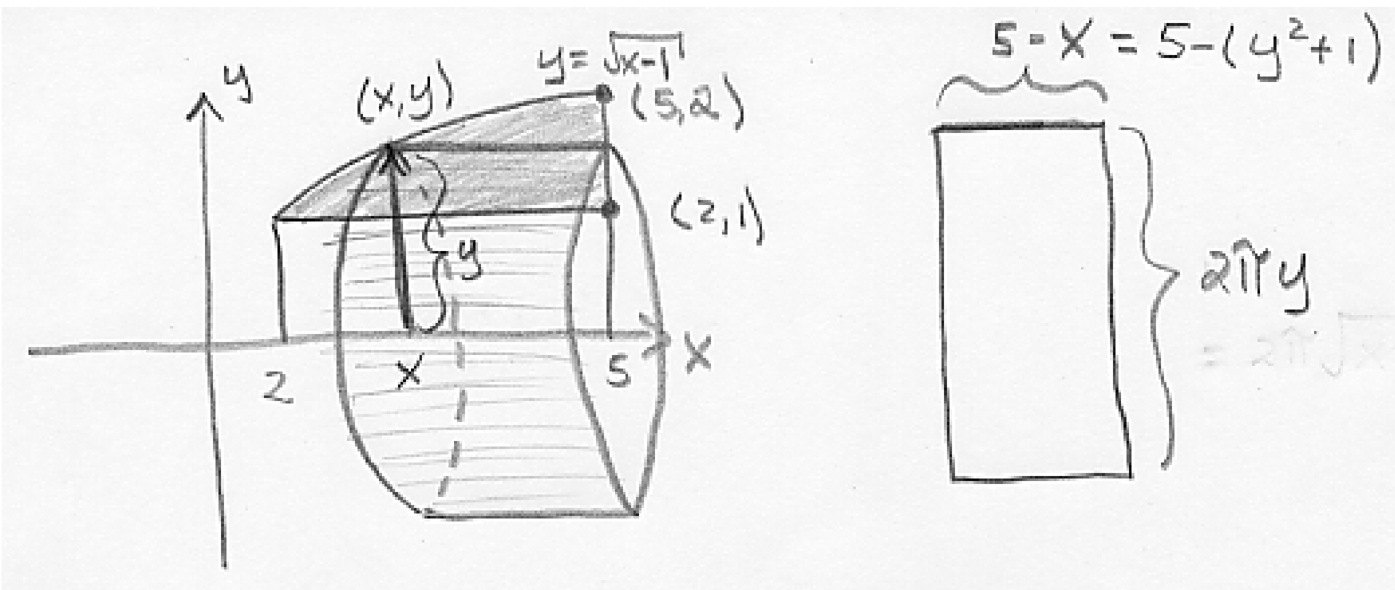


Integration limits: if $2 < x < 5$, the circle sweeps out the total volume. We integrate with respect to x.
Radius of circle $= y = \sqrt{x-1}$.

Area of circle $= \pi(\text{radius})^2 = \pi(\sqrt{x-1})^2 = \pi(x-1)$.

$$\text{Volume} = \int_a^b (\text{area of circle})\, dx = \int_2^5 \pi(x-1)\, dx = \pi \int_2^5 (x-1)\, dx = \pi \left(\frac{x^2}{2} - x\right)_2^5 = \pi \left(\frac{25}{2} - 5\right) - (2-2) = \frac{15\pi}{2}$$

Now, let's use the method of cylindrical shells. Begin with a sketch.

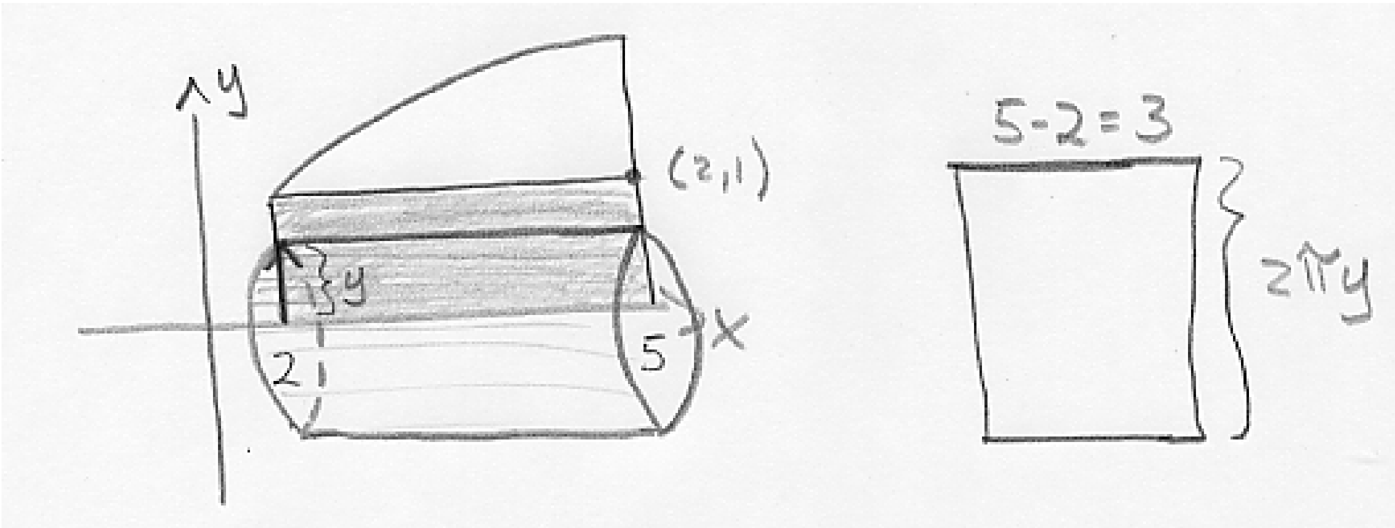


We can use cylindrical shells, but it is a bit more complicated. The height of the cylinder is <u>not</u> the same function over the integration of y. We therefore have to find the volume in two parts.

Part 1:
Integration limits: if $1 < y < 2$, the cylinder sweeps out the volume for Part 1. We integrate with respect to y.
Cylinder height $= 5 - x = 5 - y^2 - 1 = 4 - y^2$.
Cylinder radius $= y$.
Cylinder circumference $= 2\pi \times (\text{radius}) = 2\pi y$.
Cylinder surface area $= 2\pi y(4 - y^2) = 2\pi(4y - y^3)$.

$$V_1 = \int_c^d (\text{area of cylinder})\, dy = \int_1^2 2\pi(4y - y^3)\, dy = 2\pi \left(4\frac{y^2}{2} - \frac{y^4}{4}\right)_1^2 = 2\pi \left(2(2)^2 - \frac{2^4}{4}\right) - \left(2 - \frac{1}{4}\right) = \frac{9\pi}{2}$$

Part 2: Here is the sketch for the second part:



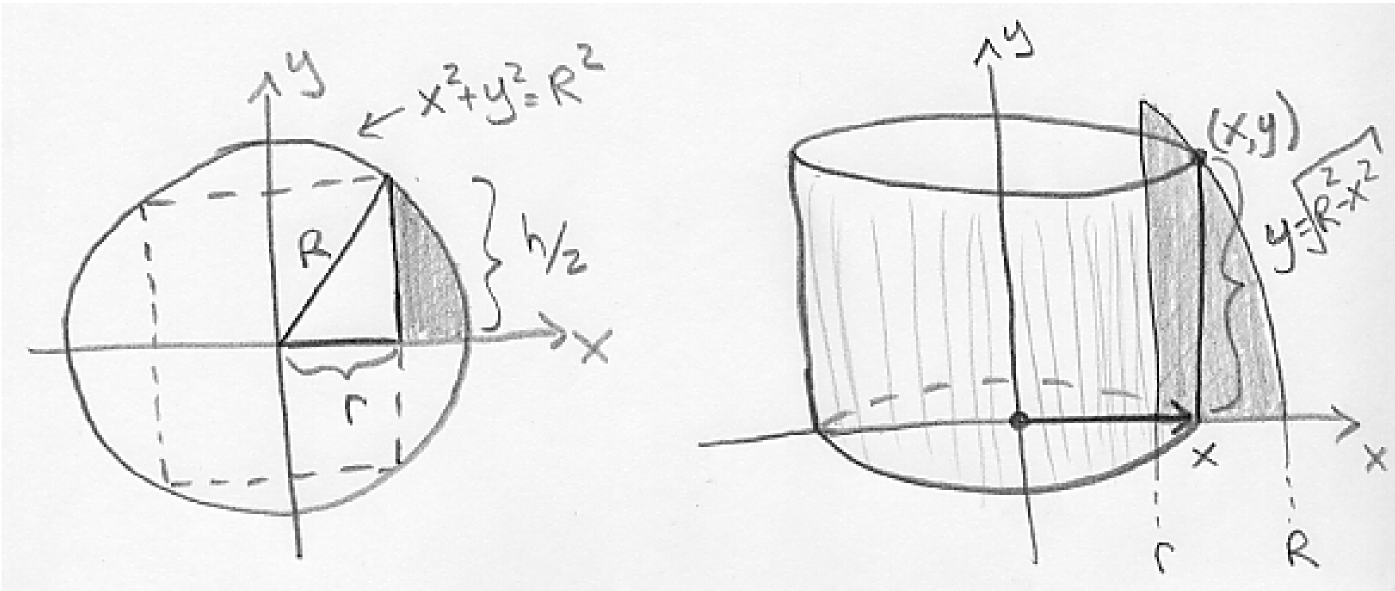
Integration limits: if $0 < y < 1$, the cylinder sweeps out the volume for Part 2. We integrate with respect to y.
Cylinder height $= 5 - 2 = 3$.
Cylinder radius $= y$.
Cylinder circumference $= 2\pi \times (\text{radius}) = 2\pi y$.
Cylinder surface area $= 2\pi y(3) = 6\pi y$.

$$V_2 = \int_c^d (\text{area of cylinder})\, dy = \int_0^1 6\pi y\, dy = 6\pi \left(\frac{y^2}{2}\right)_0^1 = 6\pi\left(\frac{1}{2}\right) = 3\pi$$

The total volume is $V_1 + V_2 = \dfrac{9\pi}{2} + 3\pi = \dfrac{15\pi}{2}$. In this case, the washer method was much easier.

Example 6.3.44 You have two spherical wooden balls of different radii, and want to make two beads. You drill a hole in one, and a hole in the other. For aesthetic reasons, the two holes have different radii. You discover that the height of both balls when you are done is the same. The mathematician inside you forces you to ask: "Which ball has more wood in it?"

Solution What you need to do is construct a diagram, and introduce some notation. You decide to work with a general ball of radius R, and drill a hole of radius r. You pick the center of the ball to be the origin of your coordinate system, with y-axis pointing along the direction of the drill hole (since you like the symmetry that introduces) and proceed.



Cylinder height $= y = \sqrt{R^2 - x^2}$.
Cylinder radius $= x$.
Cylinder circumference $= 2\pi(\text{radius}) = 2\pi x$.
Cylinder surface area $= 2\pi x\sqrt{R^2 - x^2}$.
Integration limits: if $r < x < R$, the cylinder sweeps out half the volume. We integrate with respect to x.

$$\frac{1}{2}\cdot \text{Volume} = \int_r^R 2\pi x\sqrt{R^2 - x^2}\,dx \qquad \text{Substitution: } \begin{array}{l} u = R^2 - x^2 \\ du = -2x\,dx \\ \text{change limits:} \\ \text{when } x = r \to u = R^2 - r^2 \\ \text{when } x = R \to u = 0 \end{array}$$

$$= -\pi \int_{R^2-r^2}^{0} \sqrt{u}\,du$$

$$= \pi \left.\frac{u^{3/2}}{3/2}\right|_0^{R^2-r^2} = \frac{2\pi}{3}(R^2 - r^2)^{3/2}$$

$$\text{Volume} = \frac{4\pi}{3}(R^2 - r^2)^{3/2}.$$

The geometry tells you $h^2/4 = R^2 - r^2$, so

$$\text{Volume} = \frac{4\pi}{3}(R^2 - r^2)^{3/2} = \frac{4\pi}{3}\left(\frac{h^2}{4}\right)^{3/2} = \pi h^3/6.$$

Since the volume of the bead depends only on the height of the bead, and the two beads you made have the same height, you conclude the amount of wood in the two beads is the same.

Example 6.3.18 Find the volume of the region which is created when we rotate the region bounded by the curves $y = 4x - x^2$ and $y = 8x - 2x^2$ about the axis $x = -2$.

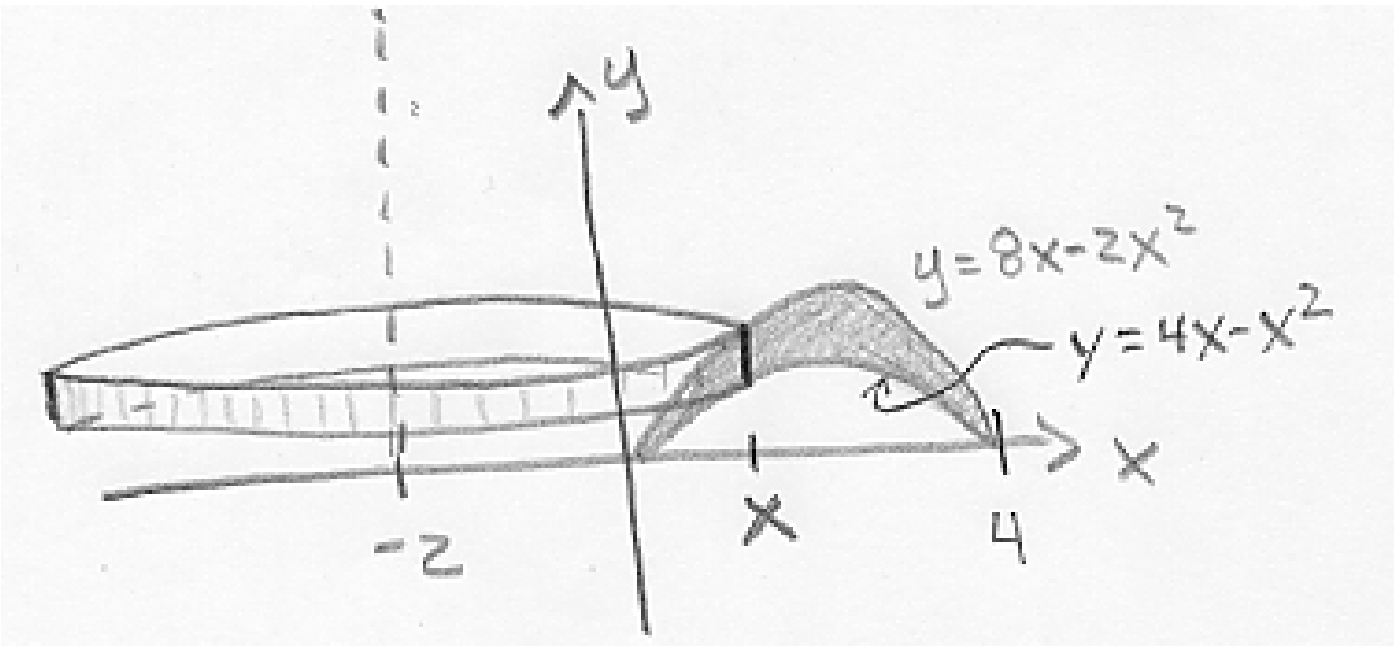
Solution Let's try the method of cylindrical shells. Begin with a sketch.

The functions are both parabolas opening down (since the squared term is negative), so we can find the roots of these equations and then get a sketch of them.

$$y = 4x - x^2 = x(4 - x)$$

$$y = 8x - 2x^2 = 2x(4 - x)$$

Both have roots at $x = 0, 4$. The function $y = 8x - 2x^2$ is twice $y = 4x - x^2$, so it will be above $y = 4x - x^2$ for $0 < x < 4$.



Integration limits: if $0 < x < 4$, the cylinder sweeps out the total volume. We integrate with respect to x.
Cylinder height $= (8x - 2x^2) - (4x - x^2) = 4x - x^2$.
Cylinder radius $= 2 + x$.
Cylinder circumference $= 2\pi \times$ (radius) $= 2\pi(x + 2)$.
Cylinder surface area $= 2\pi(x + 2)(4x - x^2) = 2\pi(8x + 2x^2 - x^3)$.

$$\text{Volume} = \int_a^b (\text{area of cylinder})\, dx = \int_0^4 2\pi(8x + 2x^2 - x^3)\, dx = 2\pi \left(4x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right)_0^4 = \frac{256\pi}{3}$$