Questions

Example $\displaystyle\int \cos x \cos^5(\sin x)\, dx$.

Example $\int \cos^4 t\, dt$.

Example $\int \cos 7\theta \cos 5\theta\, d\theta$.

Example $\int \sin^3 mx\, dx$, m is a constant.

Solutions

Example $\displaystyle\int \cos x \cos^5(\sin x)\, dx$.

$$
\begin{aligned}
\int \cos x \cos^5(\sin x)\, dx &= \int \cos^5 u\, du \\
&\quad \text{Substitution: } u = \sin x, du = \cos x\, dx \\
&= \int \cos u \cos^4 u\, du \qquad \text{(odd power, factor out a cosine)} \\
&= \int \cos u (\cos^2 u)^2\, du \\
&= \int \cos u (1 - \sin^2 u)^2\, du \\
&= \int \cos u (1 + \sin^4 u - 2\sin^2 u)\, du \\
&= \int (1 + w^4 - 2w^2)\, dw \\
&\quad \text{Substitution: } w = \sin u, dw = \cos u\, du \\
&= w + \frac{1}{5}w^5 - \frac{2}{3}w^3 + C \\
&= \sin(\sin x) + \frac{1}{5}\sin^5(\sin x) - \frac{2}{3}\sin^3(\sin x) + C
\end{aligned}
$$

Example $\int \cos^4 t\, dt$.

$$
\begin{aligned}
\int \cos^4 t\, dt &= \int (\cos^2 t)^2\, dt \qquad \text{(even power, so use 1/2-angle formula)} \\
&= \int \left(\frac{1}{2}(1 + \cos 2t)\right)^2\, dt \\
&= \frac{1}{4}\int (1 + 2\cos 2t + \cos^2 2t)\, dt \\
&= \frac{1}{4}\int dt + \frac{1}{2}\int \cos 2t\, dt + \frac{1}{4}\int \cos^2 2t\, dt
\end{aligned}
$$

Let's do each integral in turn:

$$\begin{aligned}
\frac{1}{4}\int dt &= \frac{1}{4}t + c_1 \\
\frac{1}{2}\int \cos 2t\, dt &= \frac{1}{2}\int \cos u \frac{du}{2} && \text{Substitution: } u = 2t, du = 2dt \\
&= \frac{1}{4}\int \cos u\, du \\
&= \frac{1}{4}\sin u + c_2 \\
&= \frac{1}{4}\sin 2t + c_2 \\
\frac{1}{4}\int \cos^2 2t\, dt &= \frac{1}{4}\int \frac{1}{2}(1 + \cos 4t)\, dt && \text{(even power, so use 1/2-angle formula)} \\
&= \frac{1}{8}\int dt + \frac{1}{8}\int \cos 4t\, dt && \text{Substitution: } u = 4t, du = 4dt \\
&= \frac{1}{8}t + c_3 + \frac{1}{8}\int \cos u\, \frac{du}{4} \\
&= \frac{1}{8}t + c_3 + \frac{1}{32}\sin u + c_4 \\
&= \frac{1}{8}t + c_3 + \frac{1}{32}\sin 4t + c_4
\end{aligned}$$

Combine all our results, and set $c = c_1 + c_2 + c_3 + c_4$, to get:

$$\int \cos^4 t\, dt = \frac{3t}{8} + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + c$$

Example $\int \cos 7\theta \cos 5\theta\, d\theta$.

Use the trig identity

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right] \\
\cos 7\theta \cos 5\theta &= \frac{1}{2}\left[\cos(7\theta - 5\theta) + \cos(7\theta + 5\theta)\right] \\
&= \frac{1}{2}\left[\cos 2\theta + \cos 12\theta\right] \\
\int \cos 7\theta \cos 5\theta\, d\theta &= \int \frac{1}{2}\left[\cos 2\theta + \cos 12\theta\right]\, d\theta \\
&= \frac{1}{2}\int \cos 2\theta\, d\theta + \frac{1}{2}\int \cos 12\theta\, d\theta \text{ two substitutions: } && \begin{array}{l} s = 2\theta, ds = 2d\theta \\ t = 12\theta, dt = 12d\theta \end{array} \\
&= \frac{1}{4}\int \cos s\, ds + \frac{1}{24}\int \cos t\, dt \\
&= \frac{1}{4}\sin s + \frac{1}{24}\sin t + C \\
&= \frac{1}{4}\sin 2\theta + \frac{1}{24}\sin 12\theta + C
\end{aligned}$$

Example $\int \sin^3 mx\,dx$, m is a constant.

$$\begin{aligned}
\int \sin^3 mx\,dx &= \int \sin mx \sin^2 mx\,dx && \text{(odd power of sine, so factor one out)}\\
&= \int \sin mx(1-\cos^2 mx)\,dx && \text{(insert the trig identity)}\\
&= \int (1-\cos^2 mx)\sin mx\,dx && \text{(substitution: } u=\cos mx,\ du=-m\sin mx\,dx\text{)}\\
&= -\frac{1}{m}\int (1-u^2)\,du\\
&= -\frac{1}{m}\left(u-\frac{u^3}{3}\right)+C\\
&= -\frac{1}{m}\left(\cos mx-\frac{\cos^3 mx}{3}\right)+C
\end{aligned}$$