Questions

Example $\displaystyle\int \frac{1}{x\sqrt{x^2-9}}\,dx.$

Example $\displaystyle\int_0^1 \sqrt{2x-x^2}\,dx.$

Example $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx.$

Solutions

Example $\displaystyle\int \frac{1}{x\sqrt{x^2-9}}\,dx.$

The integrand contains $\sqrt{x^2-a^2}$, so we should use the trig substitution:

$$\begin{aligned}
&x = a\sec\theta = 3\sec\theta \\
&dx = 3\sec\theta\tan\theta\,d\theta \\
&\text{where } 0 < \theta < \frac{\pi}{2} \text{ or } \pi < \theta < \frac{3\pi}{2}
\end{aligned}$$

Now, we find expressions for the components of the integrand:

$$\begin{aligned}
\sqrt{x^2-9} &= \sqrt{9\sec^2\theta - 9} \\
&= 3\sqrt{\sec^2\theta - 1} \\
&= 3\sqrt{\tan^2\theta} \\
&= 3|\tan\theta| \\
&= 3\tan\theta \text{ (since } \tan\theta > 0 \text{ in our restricted domain for } \theta\text{!)} \\
x &= 3\sec\theta
\end{aligned}$$

And now we do the integral:

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^2-9}} &= \int \frac{3\sec\theta\tan\theta\,d\theta}{(3\sec\theta)(3\tan\theta)} \\
&= \frac{1}{3}\int d\theta \\
&= \frac{1}{3}\theta + c \\
&= \frac{1}{3}\arccos\left(\frac{3}{x}\right) + c, \\
\text{or} \quad &= \frac{1}{3}\arctan\left(\frac{\sqrt{x^2-9}}{3}\right) + c, \\
\text{or} \quad &= \frac{1}{3}\arcsin\left(\frac{\sqrt{x^2-9}}{x}\right) + c,
\end{aligned}$$

We could pick any one of the last three expressions for the integral. There are other expressions for the integral as well.

If you compare this with *Mathematica*'s result, you may think you have made an error. If you use the identity $\arctan x = -\arctan(1/x) + \pi/2$, you can show the two results are the same.

Example $\displaystyle\int_0^1 \sqrt{2x - x^2}\, dx$.

The integrand does not look like any of the forms we can use trig substitution on. We must therefore modify it before we can use trig substitution.

$$\begin{aligned}
\int_0^1 \sqrt{2x - x^2}\, dx &= \int_0^1 \sqrt{x(2-x)}\, dx \\
&= \int_0^1 \sqrt{x}\sqrt{2-x}\, dx \\
&= \int_0^1 \sqrt{x}\sqrt{2-(\sqrt{x})^2}\, dx \qquad \text{Substitution: } \begin{array}{ll} u = \sqrt{x} & x = 0 \to u = 0 \\ du = \frac{1}{2}\frac{dx}{\sqrt{x}} & x = 1 \to u = 1 \end{array} \\
&= 2\int_0^1 u^2\sqrt{2-u^2}\, du
\end{aligned}$$

the integrand has a $\sqrt{a^2 - u^2}$, so we should use the trig substitution:

$$\begin{aligned}
&u = a\sin\theta = \sqrt{2}\sin\theta \\
&du = \sqrt{2}\cos\theta\, d\theta \\
&\text{where } \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}
\end{aligned}$$

Instead of back substituting later, we can change the limits of this definite integral right now:
When $u = 0$, then $\theta = \arcsin 0 = 0$.
When $u = 1$, then $\theta = \arcsin(1/\sqrt{2}) = \pi/4$.
Now, we find expressions for the components of the integrand:

$$\begin{aligned}
\sqrt{2-u^2} &= \sqrt{2 - 2\sin^2\theta} \\
&= \sqrt{2}\sqrt{1-\sin^2\theta} \\
&= \sqrt{2}\sqrt{\cos^2\theta} \\
&= \sqrt{2}|\cos\theta| \\
&= \sqrt{2}\cos\theta \text{ (since } \theta \text{ runs from 0 to } \pi/4,\ \cos\theta > 0)
\end{aligned}$$

And now we do the integral:

$$\begin{aligned}
\int_0^1 \sqrt{2x - x^2}\, dx &= 2\int_0^1 u^2\sqrt{2-u^2}\, du \\
&= 2\int_0^{\pi/4} (2\sin^2\theta)(\sqrt{2}\cos\theta)(\sqrt{2}\cos\theta\, d\theta) \\
&= 8\int_0^{\pi/4} \sin^2\theta\cos^2\theta\, d\theta
\end{aligned}$$

Now we need to use some trig identities to do this trig integral:

$$
\begin{aligned}
\int_0^1 \sqrt{2x - x^2}\, dx &= 8\int_0^{\pi/4} \sin^2\theta \cos^2\theta\, d\theta \\
&= 2\int_0^{\pi/4} (1-\cos 2\theta)(1+\cos 2\theta)\, d\theta \\
&= 2\int_0^{\pi/4} (1-\cos^2 2\theta)\, d\theta \\
&= 2\int_0^{\pi/4} d\theta - 2\int_0^{\pi/4} \cos^2 2\theta\, d\theta \\
&= 2\theta\Big|_0^{\pi/4} - \int_0^{\pi/4} (1+\cos 4\theta)\, d\theta \\
&= \pi/2 - \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \cos 4\theta\, d\theta \qquad \text{Substitution: } \begin{array}{ll} w = 4\theta & \theta = 0 \to w = 0 \\ dw = 4d\theta & \theta = \pi/4 \to w = \pi \end{array} \\
&= \pi/2 - \pi/4 - \int_0^{\pi} \cos w\, dw \\
&= \pi/4 - \sin w\Big|_0^{\pi} = \pi/4
\end{aligned}
$$

An alternate solution would involve completing the square:

$$
\begin{aligned}
2x - x^2 &= -(x^2 - 2x) \\
&= -(x^2 - 2x + 1 - 1) \\
&= -((x-1)^2 - 1) \\
&= 1 - (x-1)^2
\end{aligned}
$$

So the integral becomes:

$$
\begin{aligned}
\int_0^1 \sqrt{2x - x^2}\, dx &= \int_0^1 \sqrt{1-(x-1)^2}\, dx \qquad \text{Substitution: } \begin{array}{ll} u = x - 1 & x = 0 \to u = -1 \\ du = dx & x = 1 \to u = 0 \end{array} \\
&= \int_{-1}^0 \sqrt{1-u^2}\, du
\end{aligned}
$$

The integrand has a $\sqrt{a^2 - u^2}$, so we should use the trig substitution:

$$
\begin{aligned}
&u = a\sin\theta = \sin\theta \\
&du = \cos\theta\, d\theta \\
&\text{where } \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}
\end{aligned}
$$

Instead of back substituting later, we can change the limits of this definite integral right now:
When $u = -1$, then $\theta = \arcsin(-1) = -\pi/2$.

When $u = 0$, then $\theta = \arcsin(0) = 0$.
Now, we find expressions for the components of the integrand:

$$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

And now we do the integral:

$$\begin{aligned}
\int_0^1 \sqrt{2x-x^2}\,dx &= \int_{-1}^{0} \sqrt{1-u^2}\,du \\
&= \int_{-\pi/2}^{0} \cos\theta\ \cos\theta\,d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{0} (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{0} d\theta + \frac{1}{2}\int_{-\pi/2}^{0} \cos 2\theta\,d\theta \qquad \text{Substitution: } \begin{array}{ll} w = 2\theta & \theta = 0 \to w = 0 \\ dw = 2d\theta & \theta = -\pi/2 \to w = -\pi \end{array} \\
&= \frac{1}{2}\theta\Big|_{-\pi/2}^{0} + \frac{1}{4}\int_{-\pi}^{0} \cos w\,dw \\
&= \pi/4 + \frac{1}{4}\sin w\Big|_{-\pi}^{0} = \pi/4
\end{aligned}$$

Example $\displaystyle\int \frac{\sqrt{1+x^2}}{x}\,dx$.

First, the square root suggests that a trig substitution might help. Let's try it! Let $x = \tan\theta$, so $dx = \sec^2\theta\,d\theta$. Therefore,

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x}\,dx &= \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\ \sec^2\theta\,d\theta \\
&= \int \frac{\sec\theta}{\tan\theta}\ \sec^2\theta\,d\theta \\
&= \int \frac{\sec^3\theta}{\tan\theta}\,d\theta
\end{aligned}$$

The first time I tried this integral, I converted everything to sines and cosines, then had to make a u-substitution, then had to do partial fractions! It worked, but it was a very long path to follow. That's OK, but I think there is something shorter that will get us to our destination.

Let's factor out a secant, and use $\sec^2\theta = 1 + \tan^2\theta$ to simplify:

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x}\,dx &= \int \frac{\sec^3\theta}{\tan\theta}\,d\theta \\
&= \int \frac{\sec\theta(\sec^2\theta)}{\tan\theta}\,d\theta \\
&= \int \frac{\sec\theta(1+\tan^2\theta)}{\tan\theta}\,d\theta
\end{aligned}$$

$$= \int \frac{\sec\theta}{\tan\theta}\,d\theta + \int \sec\theta\tan\theta\,d\theta$$
$$= \int \csc\theta\,d\theta + \int \sec\theta\tan\theta\,d\theta$$

The second integral is a basic form (although, probably not that common).

$$\int \sec\theta\tan\theta\,d\theta = \sec\theta + c_1$$

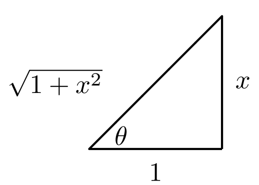
The first integral can be worked out using the same technique as was done for $\int \sec\theta\,d\theta$ in Section 7.2:

$$\begin{aligned}
\int \csc\theta\,d\theta &= \int \csc\theta\frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}\,d\theta \\
&= \int \frac{\csc^2\theta+\csc\theta\cot\theta}{\csc\theta+\cot\theta}\,d\theta \\
&\quad \text{Substitution: } u=\csc\theta+\cot\theta,\quad du=(-\csc\theta\cot\theta-\csc^2\theta)\,d\theta \\
&= -\int \frac{du}{u} \\
&= -\ln|u| + c_2 \\
&= -\ln|\csc\theta+\cot\theta| + c_2
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x}\,dx &= \int \csc\theta\,d\theta + \int \sec\theta\tan\theta\,d\theta \\
&= -\ln|\csc\theta+\cot\theta| + \sec\theta + c
\end{aligned}$$

We have used $c = c_1 + c_2$. Now, all that is left is the backsubstitution. We began with $x = \tan\theta = \text{opp}/\text{adj}$, so use that



to construct a reference triangle.

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{1+x^2}}{x}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}.$$

The integral is therefore

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x}\,dx &= -\ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + \sqrt{1+x^2} + c \\
&= -\ln\left|\frac{\sqrt{1+x^2}+1}{x}\right| + \sqrt{1+x^2} + c \\
&= \ln\left|\frac{x}{\sqrt{1+x^2}+1}\right| + \sqrt{1+x^2} + c
\end{aligned}$$