## Questions

**Example** Find an equation of the curve that passes through the point $(1,1)$ and whose slope at $(x, y)$ is $y^2/x^3$.

**Example** Find the solution of the differential equation that satisfies the initial condition (this is another way of saying initial value problem):

$$\frac{dy}{dt} = te^y, \quad y(1) = 0.$$

**Example** Find and sketch the family of curves which solve the differential equation:

$$\frac{dy}{dx} = \frac{1+x}{xy}, \quad x \neq 0.$$

**Example** Find the orthogonal trajectories of the family of curves $y = \sin x + c$.

**Example** Solve $\dfrac{dy}{dt} = \dfrac{t}{y}e^y$, $y(1) = 0$.

**Example** Solve $\dfrac{dy}{dx} = \dfrac{\ln x}{\cos^2 y}$ for $y(x)$.

## Solutions

**Example** Find an equation of the curve that passes through the point $(1,1)$ and whose slope at $(x, y)$ is $y^2/x^3$.

This is just a tricky way of asking an initial value problem. We will want to solve a differential equation, and then use the condition to determine the constant of integration, thus selecting one curve from the family of curves.

The slope of a curve is the derivative, so we need to solve

$$
\begin{aligned}
y' &= \frac{y^2}{x^3} \\
\frac{dy}{dx} &= \frac{y^2}{x^3} \\
\frac{dy}{y^2} &= \frac{dx}{x^3} \quad \text{(separate)} \\
\int \frac{dy}{y^2} &= \int \frac{dx}{x^3} \quad \text{(integrate)} \\
\int y^{-2}\, dy &= \int x^{-3}\, dx \\
\frac{y^{-1}}{-1} + c_1 &= \frac{x^{-2}}{-2} + c_2 \quad (c_1, c_2 \text{ are constants of integration}) \\
\frac{y^{-1}}{-1} &= \frac{x^{-2}}{-2} + c \quad (c = c_2 - c_1) \\
-\frac{1}{y} &= -\frac{1}{2x^2} + c \quad \text{(begin to simplify)}
\end{aligned}
$$

$$\frac{1}{y} = \frac{1}{2x^2} - \frac{2x^2 c}{2x^2}$$

$$\frac{1}{y} = \frac{1 - 2x^2 c}{2x^2}$$

$$y = \frac{2x^2}{1 - 2x^2 c}$$

So the solution to the differential equation is the family of curves

$$y(x) = \frac{2x^2}{1 - 2x^2 c}.$$

We use the condition which was supplied to pick one curve out of the family of curves. We want the curve that passes through $(1, 1)$, so we want $y(1) = 1$.

$$y(1) = 1 = \frac{2}{1 - 2c} \longrightarrow c = -\frac{1}{2}.$$

The curve which passes through the point $(1, 1)$ and has slope $y^2/x^3$ at the point $(x, y)$ is

$$y(x) = \frac{2x^2}{1 + x^2}.$$

You can verify this by taking the derivative.

**Example** Find the solution of the differential equation that satisfies the initial condition (this is another way of saying initial value problem):

$$\frac{dy}{dt} = te^y, \ \ y(1) = 0.$$

Solution:

$$\frac{dy}{dt} = te^y$$

$$e^{-y}\, dy = t\, dt \ \ \text{(separate)}$$

$$\int e^{-y}\, dy = \int t\, dt \ \ \text{(integrate)}$$

$$-e^{-y} + c_1 = \frac{t^2}{2} + c_2 \ \ (c_1, c_2 \text{ are constants of integration})$$

$$-e^{-y} = \frac{t^2}{2} + c_3 \ \ (c_3 = c_2 - c_1)$$

$$e^{-y} = -\frac{t^2}{2} - c_3 \ \ \text{(begin to simplify)}$$

$$-y = \ln\left(-\frac{t^2}{2} - c_3\right)$$

$$y = -\ln\left(-\frac{t^2}{2} - c_3\right)$$

The solution to the differential equation is the family of curves

$$y(t) = -\ln\left(-\frac{t^2}{2} - c_3\right).$$

We then use the initial condition to pick a single curve out of the family of curves,

$$y(1) = 0 = -\ln\left(-\frac{1^2}{2} - c_3\right),$$

and for the logarithm to be zero, the argument of the logarithm must be one, so

$$-\frac{1^2}{2} - c_3 = 1 \longrightarrow c_3 = -\frac{3}{2}.$$

The solution to the initial value problem is therefore

$$y(t) = -\ln\left(-\frac{t^2}{2} + \frac{3}{2}\right).$$

**Example** Find and sketch the family of curves which solve the differential equation:

$$\frac{dy}{dx} = \frac{1+x}{xy}, \quad x \neq 0.$$

We begin by solving the differential equation.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1+x}{xy} \\
y\,dy &= \frac{1+x}{x}\,dx \quad \text{(separate)} \\
\int y\,dy &= \int \frac{1+x}{x}\,dx \quad \text{(integrate)} \\
\frac{y^2}{2} + c_1 &= \int \frac{1}{x}\,dx + \int dx \\
\frac{y^2}{2} + c_1 &= \ln|x| + c_2 + x + c_3 \\
\frac{y^2}{2} &= \ln|x| + x + c \quad (c = c_2 + c_3 - c_1)
\end{aligned}
$$

This is an implicit function solution to the differential equation. Once we have loaded the ImplicitPlot command in *Mathematica*, we can plot the family of solutions using the following commands:

```
family1[x_, y_, c_] = y^2/2 == Log[Abs[x]] + x + c
ContourPlot[Evaluate[{family1[x,y,-3],family1[x,y,-1],family1[x,y,0],
   family1[x,y,1],family1[x,y,2],family1[x,y,3]}],{x,-5,5},{y,-5,5},
   AxesLabel->{"x","y"},PlotPoints->100]
```
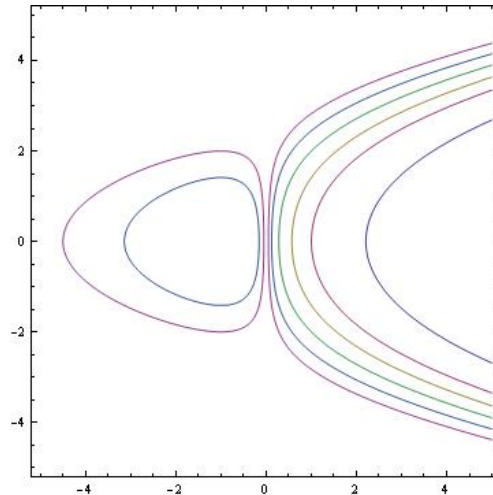
Notice the behaviour is very different for $x < 0$ and $x > 0$.

**Example** Find the orthogonal trajectories of the family of curves $y = \sin x + c$.

We begin by determining the differential equation the family of curves satisfies.

$$
\begin{aligned}
y &= \sin x + c \\
\frac{d}{dx}[y] &= \frac{d}{dx}[\sin x + c] \\
\frac{dy}{dx} &= \cos x
\end{aligned}
$$

Note that in this case the constant disappears when it is hit with the derivative, so there is no need to back substitute to eliminate the constant. Cool!

So the family of curves has slope at the point $(x, y)$ given by $y' = \cos x$. The orthogonal trajectories will have slope at the point $(x, y)$ given by $y' = -\sec x$. This is the differential equation we must solve to determine the orthogonal trajectories.

$$
\begin{aligned}
\frac{dy}{dx} &= -\sec x \\
dy &= -\sec x \, dx \\
\int dy &= -\int \sec x \, dx \\
y &= -\ln|\sec x + \tan x| + k
\end{aligned}
$$

The integral of $\sec x$ is discussed on page 486 of the text.

Here is a sketch. To be able to see the orthogonality condition clearly, we must be careful to plot equal $x$ and $y$ lengths and use an AspectRatio of 1. I've added some color to clearly distinguish the two families.

```
fam1[x_,c_]=Sin[x]+c
plot1=Plot[{fam1[x,-1],fam1[x,0],fam1[x,1]},{x,-5,5},PlotRange->{{-5,5},{-5,5}},
  PlotStyle->Blue];
```

```
fam2[x_,k_]=-Log[Abs[Sec[x]+Tan[x]]]+k
plot2=Plot[{fam2[x,-1],fam2[x,0],fam2[x,1]},{x,-5,5},PlotRange->{{-5,5},{-5,5}},
  PlotStyle->Red];

Show[plot1,plot2,AxesLabel->{"x","y"},AspectRatio->1]
```
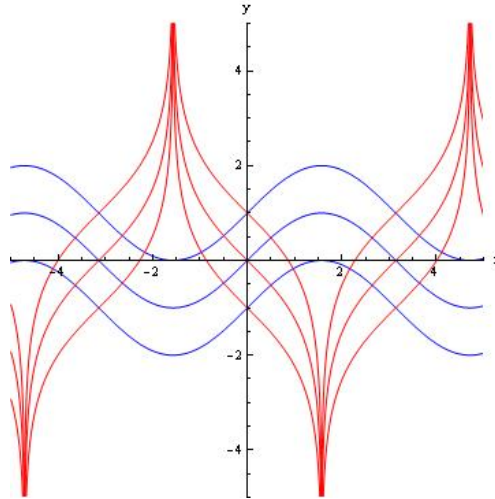


**Example** Solve $\dfrac{dy}{dt} = \dfrac{t}{y}e^y$, $y(1) = 0$.

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{t}{y}e^y \\
ye^{-y}\,dy &= t\,dt \\
\int ye^{-y}\,dy &= \int t\,dt
\end{aligned}
\tag{1}
$$

Use parts on the first integral, with $u = y, dv = e^{-y}\,dy$. Therefore, $du = dy$ and $v = -e^{-y}$.

$$
\begin{aligned}
\int ye^{-y}\,dy &= \int u\,dv \\
&= uv - \int v\,du \\
&= -ye^{-y} + \int e^{-y}\,dy \\
&= -ye^{-y} - e^{-y} + c_1 \\[6pt]
\int t\,dt &= \frac{t^2}{2} + c_2
\end{aligned}
$$

Substituting these integrals back into Equation (1), we get

$$
\begin{aligned}
-ye^{-y} - e^{-y} + c_1 &= \frac{t^2}{2} + c_2 \\
-(y+1)e^{-y} &= \frac{t^2}{2} + c
\end{aligned}
$$

This is an implicit solution to the differential equation. It would be difficult to solve for $y(t)$, so we are content with the implicit solution. The constant $c = c_2 - c_1$.

Now, use the initial condition to determine the value of the constant $c$. At $t = 1, y = 0$, so

$$
\begin{aligned}
-(0+1)e^{-0} &= \frac{1^2}{2} + c \\
-\frac{3}{2} &= c
\end{aligned}
$$

The solution to the initial value problem is $-(y+1)e^{-y} = \dfrac{t^2 - 3}{2}$.

**Example** Solve $\dfrac{dy}{dx} = \dfrac{\ln x}{\cos^2 y}$ for $y(x)$.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\ln x}{\cos^2 y} \\
\cos^2 y \, dy &= \ln x \, dx \\
\int \cos^2 y \, dy &= \int \ln x \, dx \quad\quad (2)
\end{aligned}
$$

Now do each integral in turn:

$$
\begin{aligned}
\int \cos^2 y \, dy &= \int \frac{1}{2}(1 + \cos(2y)) \, dy \\
&= \frac{1}{2} \int dy + \frac{1}{2} \int \cos(2y) \, dy \\
&= \frac{1}{2}y + \frac{1}{4}\sin(2y) + c_1
\end{aligned}
$$

The logarithm integral you may have memorized; if not work it out using parts, with $u = \ln x$, $dv = dx$. Then $du = dx/x$ and $v = x$, so

$$
\begin{aligned}
\int \ln x \, dx &= \int u \, dv \\
&= uv - \int v \, du \\
&= x \ln x - \int dx \\
&= x \ln x - x + c_2
\end{aligned}
$$

Substituting these results back into Equation ($2$) yields

$$\frac{1}{2}y + \frac{1}{4}\sin(2y) + c_1 = x\ln x - x + c_2$$

$$\frac{1}{2}y + \frac{1}{4}\sin(2y) = x\ln x - x + c$$

which is an implicit solution to the original differential equation ($c = c_2 - c_1$).