Example Find the Taylor series of $f(x) = e^{-x}$ about $x = 0$.

The center of our Taylor series will be $a = 0$. This means it could be called a MacLaurin series.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$. We will want the general form, so we should try and write things in ways in which the pattern becomes evident.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$
0	e^{-x}	1
1	$-e^{-x}$	-1
2	$+e^{-x}$	$+1$
3	$-e^{-x}$	-1
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n e^{-x}$	$(-1)^n$

So we can see that the general form is $f^{(n)}(0) = (-1)^n$, since if we take an even derivative we get a positive number, and if we take an odd derivative the number is negative.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{n!}$$

The Taylor series is given by

$$e^{-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n, \quad |x| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where $a_n = \dfrac{(-1)^n}{n!} x^n$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^n} \right| \\
&= \lim_{n\to\infty} \left| x \frac{n!}{(n+1)!} \right| \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= |x| \cdot 0 = 0 < 1 \quad \text{for all } x.
\end{aligned}$$

So the series is absolutely convergent for all values of x, which means $R = \infty$.

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n, \quad x \in (-\infty, \infty).$$

This can be checked in *Mathematica* using:

```
f[x_] = Exp[-x]
Series[f[x], {x, 0, 5}]
```

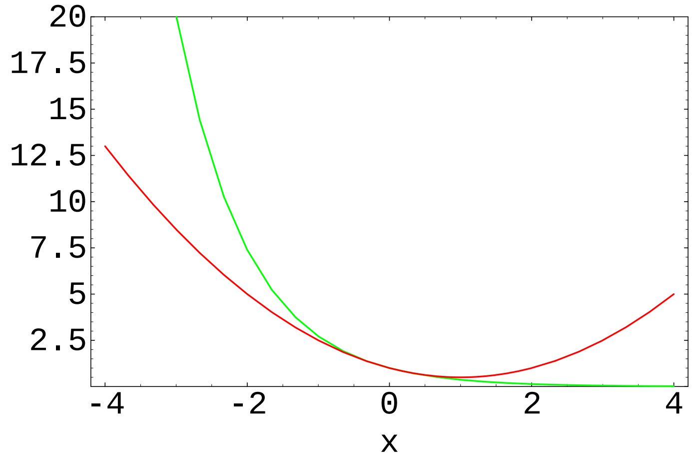

Figure 1: Plots of $f(x) = e^{-x}$ (green) and the Taylor polynomial approximation of order 2 centered at $a = 2$, $T_2(x) = \sum_{n=0}^{2} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2}$ (red).

Example Find the Taylor series of $f(x) = e^{-x}$ about $x = 3$.

The center of our Taylor series will be $a = 3$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(3)$
0	e^{-x}	$1e^{-3}$
1	$-e^{-x}$	$-1e^{-3}$
2	$+e^{-x}$	$+1e^{-3}$
3	$-e^{-x}$	$-1e^{-3}$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n e^{-x}$	$(-1)^n e^{-3}$

So we can see that the general form is $f^{(n)}(3) = (-1)^n e^{-3}$.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n}{e^3 n!}$$

The Taylor series is given by

$$e^{-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{e^3 n!}(x-3)^n, \quad |x-3| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where

$a_n = \frac{(-1)^n}{e^3 n!}(x-3)^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{e^3(n+1)!}\cdot\frac{e^3 n!}{(-1)^n(x-3)^n}\right| \\
&= \lim_{n\to\infty}\left|(x-3)\frac{n!}{(n+1)!}\right| \\
&= |x-3|\lim_{n\to\infty}\frac{1}{n+1} \\
&= |x-3|\cdot 0 = 0 < 1 \quad \text{for all } x.
\end{aligned}$$

So the series is absolutely convergent for all values of x, which means $R = \infty$.

$$e^{-x} = \sum_{n=0}^{\infty}\frac{(-1)^n}{e^3 n!}(x-3)^n, \quad x \in (-\infty, \infty).$$

This can be checked in *Mathematica* using:

```
f[x_] = Exp[-x]
Series[f[x], {x, 3, 5}]
```

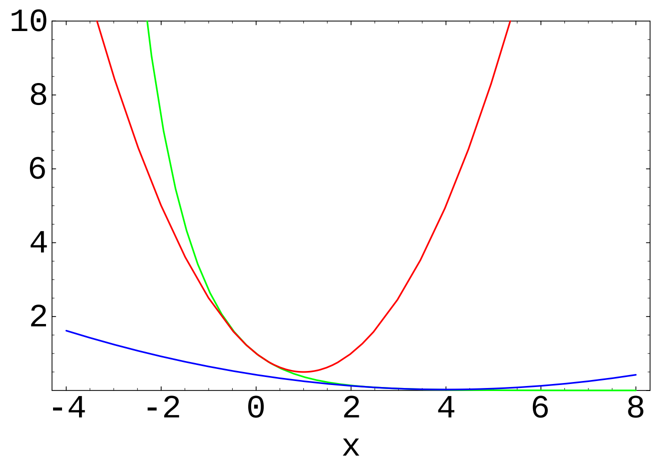

Figure 2: Plots of $f(x) = e^{-x}$ (green) and the two Taylor polynomial approximations of order 2, one centered at $a = 0$ (red) and the other centered at $a = 3$ (blue).

Example 11.10.12 Find the Taylor series of $f(x) = \ln x$ about $x = 2$.

The center of our Taylor series will be $a = 2$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$1/2$
2	$-x^{-2}$	$-1/2^2$
3	$+2x^{-3}$	$+2/2^3$
4	$-2 \cdot 3x^{-4}$	$-2 \cdot 3/2^4$
$\vdots$	$\vdots$	$\vdots$
$n \neq 0$	$(-1)^{n+1}(n-1)!\frac{1}{x^n}$	$(-1)^{n+1}(n-1)!\frac{1}{2^n}$

So we can see that the general form is $f^{(n)}(2) = (-1)^{n+1}(n-1)!\frac{1}{2^n}$ if $n \neq 0$, and $f^{(0)}(2) = \ln 2$. Since the form changes, we will have to pull the $n = 0$ term out of our sum.

$$c_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n+1}(n-1)!\frac{1}{2^n}}{n!} = \frac{(-1)^{n+1}\frac{1}{2^n}}{n}, \ \ n \neq 0; \quad c_0 = \ln 2$$

The Taylor series is given by

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n}(x-2)^n, \quad |x-2| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where $a_n = \frac{(-1)^{n+1}}{2^n n}(x-2)^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{(-1)^{n+1}(x-2)^n}\right| \\
&= \lim_{n\to\infty}\left|(x-2)^{n+1-n}2^{n-n-1} \cdot \frac{n}{n+1}\right| \\
&= \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} \\
&= \frac{|x-2|}{2}\lim_{n\to\infty}\frac{1}{1+1/n} \\
&= \frac{|x-2|}{2} \cdot \frac{1}{1+0} = \frac{|x-2|}{2} < 1
\end{aligned}$$

So the series is absolutely convergent for $|x-2| < 2$ which means $R = 2$.

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n}(x-2)^n, \quad |x-2| < 2.$$

This can be checked in *Mathematica* using:

```
f[x_] = Log[x]
Series[f[x], {x, 0, 5}]
```

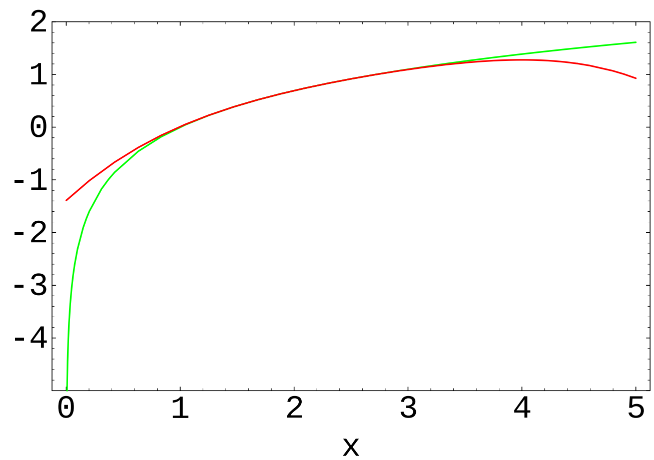

Figure 3: Plots of $f(x) = \ln x$ (green) and the Taylor polynomial approximation of order 4 centered at $a = 2$, $T_4(x) = \ln 2 + \sum_{n=1}^{4} \frac{(-1)^{n+1}}{2^n n}(x-2)^n$ (red).

Example 11.11.2 Find the Taylor series of $f(x) = 1/(1+x)^4$ about $x = 0$.

The center of our Taylor series will be $a = 0$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$
0	$(1+x)^{-4}$	1
1	$-4(1+x)^{-5}$	-4
2	$4 \cdot 5\,(1+x)^{-6}$	$+4 \cdot 5$
3	$-4 \cdot 5 \cdot 6\,(1+x)^{-7}$	$-4 \cdot 5 \cdot 6$
4	$4 \cdot 5 \cdot 6 \cdot 7\,(1+x)^{-8}$	$+4 \cdot 5 \cdot 6 \cdot 7$
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n \frac{1}{2 \cdot 3}(n+3)!(1+x)^{n+4}$	$(-1)^n \frac{(n+3)!}{6}$

So we can see that the general form is $f^{(n)}(0) = (-1)^n \frac{(n+3)!}{6}$.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n \frac{(n+3)!}{6}}{n!} = (-1)^n \frac{(n+1)(n+2)(n+3)}{6}.$$

The Taylor series is given by

$$\frac{1}{(1+x)^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)(n+3)}{6} x^n, \quad |x| < R.$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where

$a_n = (-1)^n \frac{(n+1)(n+2)(n+3)}{6} x^n.$

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+2)(n+3)(n+4)x^{n+1}}{6} \cdot \frac{6}{(-1)^n x^n (n+1)(n+2)(n+3)}\right| \\
&= \lim_{n\to\infty} \left| x^{n+1-n} \cdot \frac{(n+2)(n+3)(n+4)}{(n+1)(n+2)(n+3)} \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{(n+4)}{(n+1)} \right| \\
&= |x| \lim_{n\to\infty} \left| \frac{(1+4/n)}{(1+1/n)} \right| \\
&= |x| \cdot \frac{1+0}{1+0} = |x| < 1
\end{aligned}$$

So the series is absolutely convergent for $|x| < 1$ which means $R = 1$.

$$\frac{1}{(1+x)^4} = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)(n+3)}{6} x^n, \quad |x| < 1.$$

```
f[x_] = 1/(1+x)^4
Series[f[x], {x, 0, 5}]
```

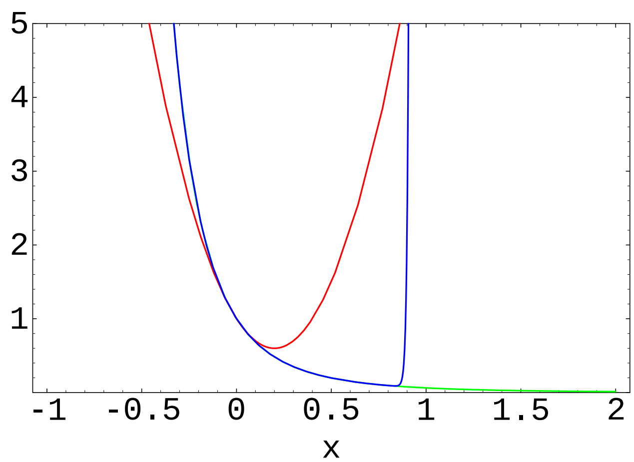

Figure 4: Plots of $f(x) = 1/(1+x)^4$ (green) and the Taylor polynomial approximation centered at $a = 0$ of order 2 (red), and 100 (blue).