Example: The Alternating Harmonic Series Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converge or diverge?

Since this is an alternating series, we should use the alternating series test. First, we identify

$$a_n = (-1)^{n-1}\frac{1}{n}, \quad b_n = \frac{1}{n}.$$

Since $1/(n+1) < 1/n$, we have that $b_{n+1} < b_n$, so the condition $b_{n+1} \leq b_n$ is satisfied.

Secondly, we have that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n = 0$.

So the two conditions of the alternating series test are satisfied, and the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges.

NOTE: The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} 1/n$ is divergent. We can prove this by the integral test.

The integral test requires that we work with $f(x)$, where
1) $f(n) = b_n$,
and on the interval $[1, \infty)$, $f(x)$ must be:
1) continuous,
2) positive,
3) decreasing.

So $f(x) = 1/x$, which is continuous, positive, and decreasing on $[1, \infty)$.

$$\begin{aligned}
\int_1^{\infty} f(x)\,dx &= \int_1^{\infty} \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \ln x|_1^b \\
&= \lim_{b\to\infty} (\ln b - \ln 1) \\
&= \lim_{b\to\infty} \ln b \\
&= \infty
\end{aligned}$$

So the integral diverges. Therefore, $\sum_{n=1}^{\infty} 1/n$ diverges by the integral theorem.

Example How many terms are required to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

to 0.001 accuracy?

First, we have to check that the series converges by the alternating series test. Then we can use the remainder estimate for the alternating series test.

Here, we have

$$a_n = \frac{(-1)^{n+1}}{n^4}, \quad b_n = \frac{1}{n^4}.$$

Since $1/(n+1)^4 < 1/n^4$, we have that $b_{n+1} < b_n$, so the condition $b_{n+1} \leq b_n$ is satisfied.

Secondly, we have that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n^4 = 0$.

So the two conditions of the alternating series test are satisfied, and the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$ converges.

The remainder estimate for the alternating series test tells us that if we approximate the series sum s by the partial sum s_n, the error will be

$$|R_n| \leq b_{n+1}.$$

n	b_n
1	1.0
2	0.0625
3	0.0123457
4	0.003906
5	0.0016
6	0.00077

Since $b_6 < 0.001$, we can say that

$$\begin{aligned} |R_n| &\leq b_{n+1} \\ |R_5| &\leq b_6 = 0.00077 \end{aligned}$$

So using the first five terms will produce an accuracy of 0.001.

Example Test the series for convergence or divergence

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}.$$

Although this doesn't initially look like an alternating series, it is an alternating series since the sine function alternates

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!} = 1 + 0 - \frac{1}{6} + 0 + \frac{1}{120} + 0 - \frac{1}{5040} + \ldots = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \ldots.$$

We therefore have

$$a_n = \frac{\sin(n\pi/2)}{n!}$$

and since

$$\begin{aligned} \sum_{i=1}^{\infty} b_n &= \sum_{i=1}^{\infty} \left| \frac{\sin(n\pi/2)}{n!} \right| = 1 + \frac{1}{6} + \frac{1}{120} + \frac{1}{5040} + \ldots \\ &= \sum_{i=1}^{\infty} \frac{1}{(2n-1)!} \\ \longrightarrow b_n &= \frac{1}{(2n-1)!} \end{aligned}$$

Since $\frac{1}{(2(n+1)-1)!} = \frac{1}{(2n+1)!} < \frac{1}{(2n-1)!}$, we have that $b_{n+1} < b_n$, so the condition $b_{n+1} \leq b_n$ is satisfied.

Secondly, we have that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/(2n-1)! = 0$.

So the two conditions of the alternating series test are satisfied, and the series $\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$ converges.