**Example 11.6.5** Is the series $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \frac{(-3)^n}{n^3}$.

The $a_n$ contains a power involving $n$, so we should try the root test.

$$\begin{aligned} \lim_{n\to\infty} (|a_n|)^{1/n} &= \lim_{n\to\infty} \left( \left| \frac{(-3)^n}{n^3} \right| \right)^{1/n} \\ &= \lim_{n\to\infty} \left( \left| \frac{3^n}{n^3} \right| \right)^{1/n} \\ &= \lim_{n\to\infty} \frac{3}{n^{3/n}} \end{aligned}$$

So we need to know what happens to $n^{3/n}$ as $n \to \infty$. This will turn out to require logarithms to solve.

$$\begin{aligned} \lim_{n\to\infty} n^{3/n} &\longrightarrow \infty^0 \quad \text{indeterminate power} \\ y &= n^{3/n} \\ \ln y &= \ln n^{3/n} = \frac{3}{n} \ln n \\ \lim_{n\to\infty} \ln y &= \lim_{n\to\infty} \frac{3}{n} \ln n \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient} \end{aligned}$$

Now we should convert to the reals, since we want to use L'Hospital's Rule to evaluate this integral.

$$\begin{aligned} \lim_{x\to\infty} \ln y &= 3 \lim_{x\to\infty} \frac{\ln x}{x} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient} \\ &= \lim_{x\to\infty} \frac{1/x}{1} \quad \text{using L'Hospital's Rule} \\ &= 0 \end{aligned}$$

We want the limit

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1$$

So, since we had constructed the real function $x^{3/x}$ from the discrete $n^{3/n}$, we can also say

$$\lim_{n\to\infty} n^{3/n} = 1.$$

Therefore, we have

$$\lim_{n\to\infty} (|a_n|)^{1/n} = \lim_{n\to\infty} \frac{3}{n^{3/n}} = \frac{3}{1} = 3 > 1$$

so the series $\sum a_n$ diverges by the root test.

**Example 11.6.7** Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{5+n}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \frac{(-1)^n}{5+n}$.

The $a_n$ is alternating, so we should try the alternating series test.

For the alternating series test, we also need to identify $b_n = |a_n| = \frac{1}{5+n}$.

Since $b_{n+1} = \frac{1}{5+n+1} = \frac{1}{6+n} < \frac{1}{5+n} = b_n$ the first condition for the alternating series test is satisfied.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5+n} = 0$, the second condition for the alternating series test is satisfied.

Therefore, by the alternating series test, the series $\sum a_n$ converges.

But we need to check the convergence of the series $\sum b_n$ to determine if the series $\sum a_n$ is conditionally convergent (that is, convergent due to the fact that it alternates).

Let's use the ratio test to check the series $\sum b_n$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{5+n}{6+n} \\
&= \lim_{n\to\infty} \frac{5/n+1}{6/n+1} \\
&= \frac{0+1}{0+1} \\
&= 1
\end{aligned}$$

so the ratio test fails. All this means is we can't use it.

Let's try a limit comparison test instead. Let's compare to the divergent $p$-series $\sum c_n = \sum 1/n$.

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{5+n}{n} = \lim_{n\to\infty} \left( \frac{5}{n} + 1 \right) = 1 > 0 \text{ and finite.}$$

Therefore, the since the comparison series $\sum c_n$ was divergent, the series $\sum b_n$ is also divergent.

Therefore, $\sum a_n$ is conditionally convergent since $\sum a_n$ converges and $\sum |a_n| = \sum b_n$ diverges.

**Example 11.6.8** Is the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \frac{(-1)^{n-1}}{n!}$.

The $a_n$ contains a factorial, so we should first try the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \\
&= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0 < 1
\end{aligned}$$

so the series $\sum a_n$ is absolutely convergent by the ratio test.

**Example 11.6.16** Is the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = (-1)^{n+1} \frac{n^2 2^n}{n!}$.

The $a_n$ contains a factorial, so we should first try the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 2^n} \\
&= \lim_{n\to\infty} \frac{2(n+1)^2}{(n+1)n^2} \\
&= 2 \lim_{n\to\infty} \frac{n+1}{n^2} \\
&= 2 \lim_{n\to\infty} \left( \frac{1}{n} + \frac{1}{n^2} \right) \\
&= 2(0+0) = 0 < 1
\end{aligned}$$

so the series $\sum a_n$ is absolutely convergent by the ratio test.

**Example 11.6.25** Is the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ absolutely convergent, conditionally convergent, or divergent?

We identify $a_n = \left(\dfrac{n^2+1}{2n^2+1}\right)^n$.

The $a_n$ contains a power involving $n$, so we should try the root test.

$$\begin{aligned}
\lim_{n\to\infty}\left(|a_n|\right)^{1/n} &= \lim_{n\to\infty}\left(\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|\right)^{1/n} \\
&= \lim_{n\to\infty}\left(\left(\frac{n^2+1}{2n^2+1}\right)^n\right)^{1/n} \\
&= \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} \\
&= \lim_{n\to\infty}\frac{1+1/n^2}{2+1/n^2} \\
&= \frac{1+0}{2+0} \\
&= \frac{1}{2} < 1
\end{aligned}$$

so the series $\sum a_n$ is absolutely convergent by the root test.