Example $\displaystyle\int \frac{1}{x\sqrt{x^2-9}}\,dx.$

The integrand contains $\sqrt{x^2-a^2}$, so we should use the trig substitution:

$$\begin{aligned}
&x = a\sec\theta = 3\sec\theta\\
&dx = 3\sec\theta\tan\theta\,d\theta\\
&\text{where } 0<\theta<\frac{\pi}{2} \text{ or } \pi<\theta<\frac{3\pi}{2}
\end{aligned}$$

Now, we find expressions for the components of the integrand:

$$\begin{aligned}
\sqrt{x^2-9} &= \sqrt{9\sec^2\theta-9}\\
&= 3\sqrt{\sec^2\theta-1}\\
&= 3\sqrt{\tan^2\theta}\\
&= 3|\tan\theta|\\
&= 3\tan\theta \text{ (since } \tan\theta>0 \text{ in our restricted domain for } \theta!)\\
x &= 3\sec\theta
\end{aligned}$$

And now we do the integral:

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-9}} &= \int \frac{3\sec\theta\tan\theta\,d\theta}{(3\sec\theta)(3\tan\theta)}\\
&= \frac{1}{3}\int d\theta\\
&= \frac{1}{3}\theta + c\\
&= \frac{1}{3}\arccos\left(\frac{3}{x}\right)+c,\\
\text{or} \quad &= \frac{1}{3}\arctan\left(\frac{\sqrt{x^2-9}}{3}\right)+c,\\
\text{or} \quad &= \frac{1}{3}\arcsin\left(\frac{\sqrt{x^2-9}}{x}\right)+c,
\end{aligned}$$

We could pick any one of the last three expressions for the integral. There are other expressions for the integral as well.

If you compare this with *Mathematica*'s result, you may think you have made an error. If you use the identity $\arctan x = -\arctan(1/x)+\pi/2$, you can show the two results are the same.

Example $\int_0^1 \sqrt{2x - x^2}\, dx$.

The integrand does not look like any of the forms we can use trig substitution on. We must therefore modify it before we can use trig substitution.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{2x-x^2}}\, dx &= \int_0^1 \frac{1}{\sqrt{x(2-x)}}\, dx \\
&= \int_0^1 \frac{1}{\sqrt{x}\sqrt{2-x}}\, dx \\
&= \int_0^1 \frac{1}{\sqrt{x}\sqrt{2-(\sqrt{x})^2}}\, dx \qquad \text{Substitution: } \begin{matrix} u = \sqrt{x} & x = 0 \to u = 0 \\ du = \frac{1}{2}\frac{dx}{\sqrt{x}} & x = 1 \to u = 1 \end{matrix} \\
&= 2\int_0^1 \frac{1}{\sqrt{2-u^2}}\, du
\end{aligned}$$

the integrand has a $\sqrt{a^2 - u^2}$, so we should use the trig substitution:

$$\begin{aligned}
&u = a\sin\theta = \sqrt{2}\sin\theta \\
&du = \sqrt{2}\cos\theta\, d\theta \\
&\text{where } \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}
\end{aligned}$$

Instead of back substituting later, we can change the limits of this definite integral right now:
When $u = 0$, then $\theta = \arcsin 0 = 0$.
When $u = 1$, then $\theta = \arcsin(1/\sqrt{2}) = \pi/4$.
Now, we find expressions for the components of the integrand:

$$\begin{aligned}
\sqrt{2-u^2} &= \sqrt{2 - 2\sin^2\theta} \\
&= \sqrt{2}\sqrt{1-\sin^2\theta} \\
&= \sqrt{2}\sqrt{\cos^2\theta} \\
&= \sqrt{2}|\cos\theta| \\
&= \sqrt{2}\cos\theta \text{ (since } \theta \text{ runs from 0 to } \pi/4,\ \cos\theta > 0)
\end{aligned}$$

And now we do the integral:

$$\begin{aligned}
\int_0^1 \sqrt{2x-x^2}\, dx &= 2\int_0^{\pi/4} \frac{\sqrt{2}\cos\theta\, d\theta}{\sqrt{2}\cos\theta} \\
&= 2\int_0^{\pi/4} d\theta = 2\theta|_0^{\pi/4} = \frac{\pi}{2}
\end{aligned}$$