Example $\int_0^3 x\sqrt{9-x^2}\,dx$.

This can be done with a u substitution. But what if you didn't notice that? Here is how it would be done using trig substitution.

The integrand contains $\sqrt{a^2-x^2}$, with $a=3$, so we should use the trig substitution:

$$\begin{aligned} x &= a\sin\theta = 3\sin\theta \\ dx &= 3\cos\theta\,d\theta \qquad \text{where } \frac{-\pi}{2} \le \theta \le \frac{\pi}{2} \end{aligned}$$

Instead of back substituting later, we can change the limits of this definite integral right now:
When $x=0$, then $\theta = \arcsin 0 = 0$.
When $x=3$, then $\theta = \arcsin(3/3) = \pi/2$.
Now, we find expressions for all the components of the integrand:

$$\begin{aligned} \sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} \\ &= \sqrt{9}\sqrt{1-\sin^2\theta} \\ &= 3\sqrt{\cos^2\theta} \\ &= 3|\cos\theta| \\ &= 3\cos\theta \text{ (since } \theta\in[-\pi/2,\pi/2],\ \cos\theta\ge 0, \text{ so } |\cos\theta| = \cos\theta) \end{aligned}$$

And now we do the integral:

$$\begin{aligned} \int_0^3 x\sqrt{9-x^2}\,dx &= \int_0^{\pi/2} (3\sin\theta)(3\cos\theta)(3\cos\theta\,d\theta) \\ &= 27\int_0^{\pi/2} \cos^2\theta\sin\theta\,d\theta \quad \text{Substitution: } \begin{array}{ll} u=\cos\theta & \theta=0 \longrightarrow u=1 \\ du = -\sin\theta\,d\theta & \theta=\pi/2 \longrightarrow u=0 \end{array} \\ &= -27\int_1^0 u^2\,du \\ &= 27\int_0^1 u^2\,du \\ &= 27\left.\frac{u^3}{3}\right|_0^1 = \frac{27}{3} = 9. \end{aligned}$$

Example $\int \frac{dt}{\sqrt{t^2-6t+13}}$

We cannot do this integral directly with u substitution. The square root tells us that maybe we should try a trig substitution, but the root is not in one of the three forms we know how to treat.

First, we must complete the square:

$$t^t - 6t + 13 = (t-3)^2 - 9 + 13 = (t-3)^2 + 4$$

If you are unsure about having done this step properly, square out your result and make sure it is correct.

We now have

$$\int \frac{dt}{\sqrt{t^2 - 6t + 13}} = \int \frac{dt}{\sqrt{(t-3)^2 + 4}} \quad \text{Substitution: } \begin{array}{l} x = t - 3 \\ dx = dt \end{array}$$
$$= \int \frac{dx}{\sqrt{x^2 + 4}}$$

Since this is an indefinite integral, we need to use different substitution variables at each step and then at the end "fold up" our substitutions to get the integral in terms of the original variable t.

The integral contains a $\sqrt{x^2 + a^2}$, with $a = 2$, so we should use the trig substitution:

$$\begin{array}{l} x = a\tan\theta = 2\tan\theta \\ dx = 2\sec^2\theta \, d\theta \qquad \text{where } \frac{-\pi}{2} < \theta < \frac{\pi}{2} \end{array}$$

Now, we find expressions for all the components of the integrand:

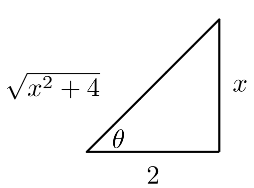
$$\begin{aligned} \sqrt{x^2 + 4} &= \sqrt{4\tan^2\theta + 4} \\ &= \sqrt{4}\sqrt{\tan^2\theta + 1} \\ &= 2\sqrt{\sec^2\theta} \\ &= 2|\sec\theta| \\ &= 2\sec\theta \text{ (since } \theta \in (-\pi/2, \pi/2), \sec\theta > 0, \text{ so } |\sec\theta| = \sec\theta) \end{aligned}$$

And now we do the integral:

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 + 4}} &= \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta} \\ &= \int \sec\theta \, d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \end{aligned}$$

We now need to back substitute for θ, and then for x, to get the final answer in terms of the original variable t. First, construct the diagram that will help us back substitute the θ:

$$\tan\theta = \frac{x}{2}$$



$$\tan\theta = \frac{x}{2}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{2/\sqrt{x^2 + 4}} = \frac{\sqrt{x^2 + 4}}{2}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+4}} &= \ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|+C \\ &= \ln\left|\frac{\sqrt{(t-3)^2+4}}{2}+\frac{t-3}{2}\right|+C\end{aligned}$$

Example $\displaystyle\int \frac{x^2}{\sqrt{4x-x^2}}\,dx$

We cannot do this integral directly with u substitution. The square root tells us that maybe we should try a trig substitution, but the root is not in one of the three forms we know how to treat.

First, we must complete the square:

$$4x-x^2 = -(x^2-4x) = -((x-2)^2-4) = 4-(x-2)^2$$

If you are unsure about having done this step properly, square out your result and make sure it is correct.

We now have

$$\begin{aligned}\int \frac{x^2}{\sqrt{4x-x^2}}\,dx &= \int \frac{x^2}{\sqrt{4-(x-2)^2}}\,dx \quad \text{Substitution: } \begin{array}{l} t=x-2 \\ dt=dx \end{array} \\ &= \int \frac{(t+2)^2}{\sqrt{4-t^2}}\,dt\end{aligned}$$

Since this is an indefinite integral, we need to use different substitution variables at each step and then at the end "fold up" our substitutions to get the integral in terms of the original variable t.

The integral contains a $\sqrt{a^2-t^2}$, with $a=2$, so we should use the trig substitution:

$$\begin{aligned}t &= a\sin\theta = 2\sin\theta \\ dt &= 2\cos\theta\,d\theta \qquad \text{where } \frac{-\pi}{2}\le\theta\le\frac{\pi}{2}\end{aligned}$$

Now, we find expressions for all the components of the integrand:

$$\begin{aligned}\sqrt{4-t^2} &= \sqrt{4-4\sin^2\theta} \\ &= \sqrt{4}\sqrt{1-\sin^2\theta} \\ &= 2\sqrt{\cos^2\theta} \\ &= 2|\cos\theta| \\ &= 2\cos\theta \text{ (since } \theta\in[-\pi/2,\pi/2],\ \cos\theta\ge 0, \text{ so } |\cos\theta|=\cos\theta) \\ (t+2)^2 &= (2\sin\theta+2)^2\end{aligned}$$

And now we do the integral:

$$\int \frac{(t+2)^2}{\sqrt{4-t^2}}\,dt = \int \frac{(2\sin\theta+2)^2}{2\cos\theta}(2\cos\theta\,d\theta)$$

$$\begin{aligned}
&= \int (2\sin\theta + 2)^2 \, d\theta \\
&= \int (4\sin^2\theta + 8\sin\theta + 4) \, d\theta \\
&= 4\int \sin^2\theta \, d\theta + 8\int \sin\theta \, d\theta + 4\int \, d\theta
\end{aligned}$$

Let's do each integral in turn. For the first integral, we have an even power of sine, so we should use the 1/2-angle formula trig identity $\sin^2\alpha = \frac{1}{2}(1 - \cos 2\alpha)$ to do the integral.

$$\begin{aligned}
4\int \sin^2\theta \, d\theta &= 4\int \frac{1}{2}(1 - \cos 2\theta) \, d\theta \\
&= 2\int \, d\theta - 2\int \cos 2\theta \, d\theta \quad \text{Substitution: } \begin{array}{c} z = 2\theta \\ dz = 2d\theta \end{array} \\
&= 2\theta - \int \cos z \, dz \\
&= 2\theta - \sin z \\
&= 2\theta - \sin 2\theta \\
&= 2\theta - 2\sin\theta\cos\theta
\end{aligned}$$

In the last line we wrote everything in terms of the angle θ rather than 2θ, since we will need to back substitute for θ later. Also, I will add a constant of integration later, once all the integrals are done.

The other two integrals are much easier to do:

$$\begin{aligned}
8\int \sin\theta \, d\theta &= 8(-\cos\theta) = -8\cos\theta \\
4\int \, d\theta &= 4\theta
\end{aligned}$$

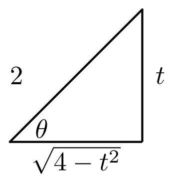
Collecting all these integrals together, and adding the constant of integration, we arrive at:

$$\int \frac{x^2}{\sqrt{4x - x^2}} \, dx = 2\theta - 2\sin\theta\cos\theta - 8\cos\theta + 4\theta + C = 6\theta - 2\sin\theta\cos\theta - 8\cos\theta + C$$

All that is left is the back substitution to get the final result in terms of x, the original variable.

First, construct the diagram that will help us back substitute the θ:

$$\sin\theta = \frac{t}{2}$$



$$\sin\theta = \frac{t}{2}, \quad \cos\theta = \frac{\sqrt{4-t^2}}{2}, \quad \theta = \arcsin\left(\frac{t}{2}\right)$$

And now we can finish the back substitution:

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4x-x^2}}\,dx &= 6\theta - 2\sin\theta\cos\theta - 8\cos\theta + 4\theta + C \\
&= 6\arcsin\left(\frac{t}{2}\right) - 2\cdot\frac{t}{2}\cdot\frac{\sqrt{4-t^2}}{2} - 8\frac{\sqrt{4-t^2}}{2} + C \\
&= 6\arcsin\left(\frac{x-2}{2}\right) - (x-2)\cdot\frac{\sqrt{4-(x-2)^2}}{2} - 4\sqrt{4-(x-2)^2} + C \\
&= 6\arcsin\left(\frac{x-2}{2}\right) - \left(\frac{x}{2}+3\right)\sqrt{4-(x-2)^2} + C
\end{aligned}$$