Other solutions than the ones presented here are possible!

Questions

1. $\displaystyle\int x \arcsin x \, dx$

2. $\displaystyle\int_0^{\pi/4} \cos^2\theta \tan^2\theta \, d\theta$

3. $\displaystyle\int \sin^2 x \cos^3 x \, dx$

4. $\displaystyle\int \sin 4x \cos 3x \, dx$

5. **(Challenging)** The functions $y = e^{x^2}$ and $y = x^2 e^{x^2}$ do not have elementary antiderivatives, but $y = (2x^2+1)e^{x^2}$ does. Evaluate $\displaystyle\int (2x^2+1)e^{x^2}\, dx$.

6. **(long)** $\displaystyle\int \frac{x}{(x-3)(x^2+4x+5)}\, dx$. This problem is a real test of our organizational abilities!

Solutions

Example 1 $\displaystyle\int x \arcsin x \, dx$

Use parts, since part of integrand is x which simplifies when differntiated:

$$\begin{aligned} u &= x & dv &= \arcsin x \, dx \\ du &= dx & v &= \int \arcsin x \, dx \end{aligned}$$

Use parts to get $v = I = \int \arcsin x \, dx$. I will reuse u and v for this part. They have no relation to u and v above.

$$\begin{aligned} u &= \arcsin x & dv &= dx \\ du &= \frac{dx}{\sqrt{1-x^2}} & v &= x \end{aligned}$$

$$
\begin{aligned}
I = \int \arcsin x \, dx &= \int u \, dv \\
&= uv - \int v \, du \quad \text{(parts)} \\
&= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx \quad \text{Substitution: } \begin{array}{l} t = 1 - x^2 \\ dt = -2x \, dx \end{array} \\
&= x \arcsin x + \frac{1}{2} \int \frac{dt}{\sqrt{t}} \\
&= x \arcsin x + \frac{1}{2} \int t^{-1/2} dt \\
&= x \arcsin x + t^{1/2} \\
&= x \arcsin x + \sqrt{1-x^2}
\end{aligned}
$$

So, we have for the first invocation of parts $v = x \arcsin x + \sqrt{1-x^2}$. To recap what we have,

$$
\begin{array}{rl}
u = x & dv = \arcsin x \, dx \\
du = dx & v = x \arcsin x + \sqrt{1-x^2}
\end{array}
$$

$$
\begin{aligned}
\int x \arcsin x \, dx &= \int u \, dv \\
&= uv - \int v \, du \\
\int x \arcsin x \, dx &= x^2 \arcsin x + x\sqrt{1-x^2} - \int x \arcsin x \, dx - \int \sqrt{1-x^2} \, dx
\end{aligned}
$$

The integral we seek appears on both sides of the equation. We can algebraically solve for it:

$$
\int x \arcsin x \, dx = \frac{1}{2}(x^2 \arcsin x + x\sqrt{1-x^2} - \int \sqrt{1-x^2} \, dx)
$$

Now all we need is the integral $\int \sqrt{1-x^2} \, dx$. For this, we should try a trig substitution. Since the integrand contains $\sqrt{a^2 - x^2}$, $a = 1$, we should use the trig substitution:

$$
\begin{aligned}
x &= a \sin\theta = \sin\theta \\
dx &= \cos\theta \, d\theta
\end{aligned}
$$

Now, we find expressions for all the components of the integrand:

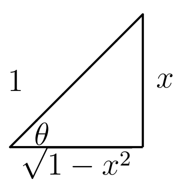
$$
\begin{aligned}
\sqrt{1-x^2} &= \sqrt{1 - \sin^2\theta} \\
&= \sqrt{\cos^2\theta} \\
&= |\cos\theta| = \cos\theta
\end{aligned}
$$

And now we do the integral:

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \int \cos\theta\cos\theta\,d\theta \\
&= \int \cos^2\theta\,d\theta \quad \text{(use 1/2-angle trig identity)} \\
&= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta\,d\theta \quad \text{Substitution: } \begin{array}{l} s = 2\theta \\ ds = 2d\theta \end{array} \\
&= \frac{1}{2}\theta + \frac{1}{4}\int \cos s\,ds \\
&= \frac{\theta}{2} + \frac{1}{4}\sin s + C \\
&= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C \\
&= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + C
\end{aligned}$$

We now need to back substitute for θ, to get the final answer in terms of the original variable x. First, construct the diagram that will help us back substitute the θ:

$$\sin\theta = \frac{x}{1}$$



$$\sin\theta = x, \quad \theta = \arcsin x, \quad , \cos\theta = \sqrt{1-x^2}$$

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + c \\
&= \frac{\arcsin x}{2} + \frac{1}{2}x\sqrt{1-x^2} + c
\end{aligned}$$

And, finally, we arrive at the result for the original integral:

$$\begin{aligned}
\int x\arcsin x\,dx &= \frac{1}{2}\left(x^2\arcsin x + x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx\right) \\
&= \frac{1}{2}\left(x^2\arcsin x + x\sqrt{1-x^2} - \frac{\arcsin x}{2} - \frac{1}{2}x\sqrt{1-x^2}\right) + C \\
&= \frac{x^2\arcsin x}{2} - \frac{\arcsin x}{4} + \frac{1}{4}x\sqrt{1-x^2} + C
\end{aligned}$$

Example 2 $\int_0^{\pi/4} \cos^2\theta \tan^2\theta \, d\theta$

$$
\begin{aligned}
\int_0^{\pi/4} \cos^2\theta \tan^2\theta \, d\theta &= \int_0^{\pi/4} \cos^2\theta \frac{\sin^2\theta}{\cos^2\theta} \, d\theta \\
&= \int_0^{\pi/4} \sin^2\theta \, d\theta \\
&= \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta) \, d\theta \\
&= \frac{1}{2}\int_0^{\pi/4} d\theta - \frac{1}{2}\int_0^{\pi/4} \cos 2\theta \, d\theta \quad \text{Substitution: } \begin{array}{l} u = 2\theta \\ du = 2d\theta \\ \theta = 0 \longrightarrow u = 0 \\ \theta = \pi/4 \longrightarrow u = \pi/2 \end{array} \\
&= \frac{1}{2}\theta\Big|_0^{\pi/4} - \frac{1}{4}\int_0^{\pi/2} \cos u \, du \\
&= \frac{\pi}{8} - \frac{1}{4}\sin u\Big|_0^{\pi/2} \\
&= \frac{\pi}{8} - \frac{1}{4}(\sin\pi/2 - \sin 0) \\
&= \frac{\pi}{8} - \frac{1}{4}
\end{aligned}
$$

Example 3 $\int \sin^2 x \cos^3 x \, dx$

$$
\begin{aligned}
\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos x \cos^2 x \, dx \\
&= \int \sin^2 x \cos x (1 - \sin^2 x) \, dx \\
&= \int (\sin^2 x - \sin^4 x) \cos x \, dx \quad \text{Substitution: } \begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array} \\
&= \int (u^2 - u^4) du \\
&= \frac{u^3}{3} - \frac{u^5}{5} + C \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C
\end{aligned}
$$

Example 4 $\int \sin 4x \cos 3x \, dx$

Use the trig identity

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin 4x \cos 3x = \frac{1}{2}[\sin(4x - 3x) + \sin(4x + 3x)] = \frac{1}{2}[\sin x + \sin 7x]$$

$$\begin{aligned}
\int \sin 4x \cos 3x \, dx &= \int \frac{1}{2}[\sin x + \sin 7x] \, dx \\
&= -\frac{1}{2}\cos x + \frac{1}{2}\int \sin(7x)\, dx \quad \text{Substitution: } \begin{array}{l} u = 7x \\ du = 7\, dx \end{array} \\
&= -\frac{1}{2}\cos x + \frac{1}{14}\int \sin u \, du \\
&= -\frac{1}{2}\cos x - \frac{1}{14}\cos u + C \\
&= -\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C
\end{aligned}$$

Example 5 (Challenging) The functions $y = e^{x^2}$ and $y = x^2 e^{x^2}$ do not have elementary antiderivatives, but $y = (2x^2 + 1)e^{x^2}$ does. Evaluate $\int (2x^2 + 1)e^{x^2}\, dx$.

The basic difficulty is evaluating $\int e^{x^2}\, dx$. So let's isolate that, and see what happens to the rest of the integrand:

$$\int (2x^2 + 1)e^{x^2}\, dx = 2\int x^2 e^{x^2}\, dx + \int e^{x^2}\, dx$$

Use parts on the first integral: Choose $u = x^2$, since it will simplify when we differentiate it. Therefore, $du = 2x dx$. What is left in the integrand is $dv = e^{x^2}\, dx$. Well, we cannot integrate this to determine v. This approach was not beneficial!

Before we abandon parts, we should think of other ways to break the integrand up. We could try $u = x$, which leads to $du = dx$, and $dv = xe^{x^2}\, dx$. This means we need to determine $v = \int xe^{x^2}\, dx$. Hey–we can do this integral using u-substitution! Since we are able to travel a bit further down this path, maybe we are on to something (we hope!).

$$\begin{aligned}
v = \int xe^{x^2}\, dx &= \frac{1}{2}\int e^w \, dw \\
&\quad \text{Substitution: } w = x^2, \; dw = 2x\, dx \\
&= \frac{1}{2}e^w \\
&= \frac{1}{2}e^{x^2}
\end{aligned}$$

Returning to the original invocation of parts, we have

$$\begin{aligned}
\int (2x^2+1)e^{x^2}\,dx &= 2\int x^2e^{x^2}\,dx + \int e^{x^2}\,dx \\
&= 2\left(\int u\,dv\right) + \int e^{x^2}\,dx \\
&= 2\left(uv - \int v\,du\right) + \int e^{x^2}\,dx \\
&= 2\left(x\cdot\frac{1}{2}e^{x^2} - \int \frac{1}{2}e^{x^2}\,dx\right) + \int e^{x^2}\,dx \\
&= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \\
&= xe^{x^2} + c
\end{aligned}$$

We were able to do this integral by cancelling out the component that we could not do! That is awesome.

We added the constant of integration at the end since we know that an indefinite integral should be determined up to a constant of integration, and for convenience we left the constant out when we did parts. If we had included a constant there, we wouldn't need to add one at the end. Remember, if you are ever concerned about constants, put them in when a definite integral occurs because it is never wrong to include them. You can only get in trouble with constants by leaving them out.

Example 6 (long) $\displaystyle\int \frac{x}{(x-3)(x^2+4x+5)}\,dx$. This problem is a real test of our organizational abilities!

If you tried u-substitution, you would choose $u = x^2+4x+5$, so $du = (2x+4)\,dx$. This does not help you with this integral.

Since we have a rational integrand, we should try partial fractions and see what happens.

$$\begin{aligned}
\frac{x}{(x-3)(x^2+4x+5)} &= \frac{A}{x-3} + \frac{Bx+C}{x^2+4x+5} \quad \text{(split)} \\
\left[\frac{x}{(x-3)(x^2+4x+5)}\right. &= \left.\frac{A}{x-3} + \frac{Bx+C}{x^2+4x+5}\right](x-3)(x^2+4x+5) \quad \text{(clear fractions)} \\
x &= A(x^2+4x+5) + (Bx+C)(x-3) \quad \text{(simplify)} \\
0 &= -x + Ax^2 + 4Ax + 5A + Bx^2 + Cx - 3Bx - 3C \quad \text{(collect powers of } x\text{)} \\
0 &= (5A-3C)x^0 + (-1+4A+C-3B)x^1 + (A+B)x^3 \quad \text{(collect powers of } x\text{)}
\end{aligned}$$

For this to be true for all values of x, we must have the coefficients of the powers of x equal to zero. Then it won't matter what we put in for x, the equation will be satisfied. So we must solve the three equations in the three unknowns A, B, C:

$$\begin{aligned}
5A - 3C &= 0 \qquad (1) \\
-1 + 4A + C - 3B &= 0 \qquad (2) \\
A + B &= 0 \qquad (3)
\end{aligned}$$

From (1) $A = -B$, and from (2) $C = 5A/3 = -5B/3$; therefore, (3) becomes $-1 + 4(-B) - 5B/3 - 3B = 0$, which means $B = -3/26$. Therefore, $A = 3/26$ and $C = 5/26$.

We have shown that

$$\frac{x}{(x-3)(x^2+4x+5)} = \frac{1}{26}\left[\frac{3}{x-3} + \frac{-3x+5}{x^2+4x+5}\right]$$

Our integral becomes:

$$\int \frac{x}{(x-3)(x^2+4x+5)}\,dx = \frac{1}{26}\left[\int \frac{3}{x-3}\,dx + \int \frac{-3x+5}{x^2+4x+5}\,dx\right] \quad (4)$$

$$\begin{aligned}
\int \frac{3}{x-3}\,dx &= 3\int \frac{du}{u} \\
&\quad \text{Substitution: } u = x-3, du = dx \\
&= 3\ln|u| + c_1 \\
&= 3\ln|x-3| + c_1 \quad (5)
\end{aligned}$$

The second integral is more complicated, however, these more complicated integrals that arise in partial fractions typically follow the same path to solution. We need to complete the square in the denominator, and then we will get a logarithm and arctangent when we integrate.

Complete the square:

$$\begin{aligned}
x^2+4x &= x^2+4x+4-4 = (x^2+4x+4)-4 = (x+2)^2-4 \\
x^2+4x+5 &= (x+2)^2+1
\end{aligned}$$

Substitute back into the integral:

$$\begin{aligned}
\int \frac{-3x+5}{x^2+4x+5}\,dx &= \int \frac{-3x+5}{(x+2)^2+1}\,dx \\
&= -3\int \frac{x}{(x+2)^2+1}\,dx + 5\int \frac{1}{(x+2)^2+1}\,dx \quad (6)
\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{(x+2)^2+1}\,dx & \\
&\quad \text{Substitution: } u = (x+2)^2+1, du = (2x+4)\,dx \\
&= \frac{1}{2}\int \frac{2x}{(x+2)^2+1}\,dx + \frac{1}{2}\int \frac{4}{(x+2)^2+1}\,dx - \frac{1}{2}\int \frac{4}{(x+2)^2+1}\,dx \\
&= \frac{1}{2}\int \frac{2x+4}{(x+2)^2+1}\,dx - \int \frac{2}{(x+2)^2+1}\,dx \\
&= \frac{1}{2}\int \frac{du}{u} - \int \frac{2}{(x+2)^2+1}\,dx \\
&= \frac{1}{2}\ln|u| - \int \frac{2}{(x+2)^2+1}\,dx + c_2 \\
&= \frac{1}{2}\ln\left|(x+2)^2+1\right| - \int \frac{2}{(x+2)^2+1}\,dx + c_2 \quad (7)
\end{aligned}$$

Whew, there is lots going on here! But we are making progress. Substitute (7) back into (6):

$$\begin{aligned}
\int \frac{-3x+5}{x^2+4x+5}\,dx &= -3\left(\frac{1}{2}\ln\left|(x+2)^2+1\right| - \int \frac{2}{(x+2)^2+1}\,dx + c_2\right) + 5\int \frac{1}{(x+2)^2+1}\,dx \\
&= -\frac{3}{2}\ln\left|(x+2)^2+1\right| + 6\int \frac{1}{(x+2)^2+1}\,dx + c_2 + 5\int \frac{1}{(x+2)^2+1}\,dx \\
&= -\frac{3}{2}\ln\left|(x+2)^2+1\right| + 11\int \frac{1}{(x+2)^2+1}\,dx + c_2 \\
&\quad \text{Substitution: } u = x+2, du = dx \\
&= -\frac{3}{2}\ln\left|(x+2)^2+1\right| + 11\int \frac{1}{u^2+1}\,du + c_2
\end{aligned}$$

and this remaining integral is a basic form, $\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\arctan(u/a) + c_3$, so with $a = 1$ we have

$$\int \frac{-3x+5}{x^2+4x+5}\,dx = -\frac{3}{2}\ln\left|(x+2)^2+1\right| + 11\arctan(x+2) + c_2 + c_3 \tag{8}$$

Finally, substituting (8) and (5) into (4) we get:

$$\int \frac{x}{(x-3)(x^2+4x+5)}\,dx = \frac{3}{26}\ln|x-3| - \frac{3}{52}\ln\left|(x+2)^2+1\right| + \frac{11}{26}\arctan(x+2) + c$$

where we have created the new constant $c = c_1 + c_2 + c_3$.