1102 Calculus II 7.5 Strategy for Integration

To be able to do integration successfully, you need some basic skills.

First, know the basic differentiation and integration formulas. These are found in the back of the textbook.

Differentiation Rules: 1–24.
Integration Formulas: 1–18.

We are not using any hyberbolic functions in this course, so you can ignore any reference to sinh, cosh, etc.

Second, you need to know some trig identities. Most of the ones you need can be easily worked out from the following three identities:

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \end{aligned}$$

Here's a more complete list:

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos 2x &= \cos^2 x - \sin^2 x \\ \cos 2x &= 1 - 2\sin^2 x \\ \cos^2 x &= \frac{1}{2}(1+\cos 2x) \\ \sin^2 x &= \frac{1}{2}(1-\cos 2x) \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(2x) &= 2 \sin x \cos x \end{aligned}$$

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ 1 + \tan^2 x &= \sec^2 x \\ \cot^2 x + 1 &= \csc^2 x \\ \sin x \cos y &= \frac{1}{2}[\sin(x+y) + \sin(x-y)] \\ \cos x \cos y &= \frac{1}{2}[\cos(x+y) + \cos(x-y)] \\ \sin x \sin y &= \frac{1}{2}[\cos(x-y) - \cos(x+y)] \end{aligned}$$

You also need to know SOH CAH TOA, Pythagoras' Theorem, and how to use reference triangles to simplify expressions.

With those mathematical concepts in hand, you are ready to tackle more difficult integrals. The idea is to use the techniques we have learned to transform an integral into one of the basic forms, which you can do easily.

Here is a procedure that you might find useful when you encounter an integral.

Step 1 Try u-substitution. Sometimes a difficult looking integrals can be conquered with this simple technique.

$$\int \frac{36x^3 - 2}{9x^4 - 2x + 45}\, dx$$

This is a ratio of polynomials, and it can be done using the substitution $u = 9x^4 - 2x + 45,\ du = (36x^3 - 2)dx$. Other examples include

$$\int \cos x \sin^5 x\, dx, \quad u = \sin x,\ du = \cos x\, dx$$

$$\int \frac{x}{\sqrt{9 - x^2}}\, dx, \quad u = 9 - x^2,\ du = -2x\, dx$$

Step 2 Simplify the integrand. This can be done by factoring, completing the square, or using trig identities. Examples of this include

$$\int \tan x \cos x\, dx = \int \frac{\sin x}{\cos x} \cos x\, dx = \int \sin x\, dx \quad \text{(sine form)}$$

$$\int \frac{t+a}{t^2-a^2}\, dt = \int \frac{t+a}{(t+a)(t-a)}\, dt = \int \frac{dt}{t-a} \quad \text{(log form)}$$

$$\int \frac{1}{4y^2-4y-3}\, dy = \int \frac{1}{(2y-1)^2-4}\, dy \quad \text{(complete square, then substitution, then partial fractions)}$$

Step 3 Classify according to the form of the integrand.

- radical $\longrightarrow$ trig substitution.

  $\sqrt{x^2-a^2} \longrightarrow x = a\sec\theta,\ dx = a\sec\theta\tan\theta\, d\theta$ (use $\tan^2\theta = \sec^2\theta - 1$)

  $\sqrt{x^2+a^2} \longrightarrow x = a\tan\theta,\ dx = a\sec^2\theta\, d\theta$ (use $\sec^2\theta = \tan^2\theta + 1$)

  $\sqrt{a^2-x^2} \longrightarrow x = a\sin\theta,\ dx = a\cos\theta\, d\theta$ (use $\cos^2\theta = 1 - \sin^2\theta$)

- trig functions $\longrightarrow$ trig identities.

  $\cos^m x \sin^n x$ either of sine and cosine have odd power, save one, use $\cos^2 x + \sin^2 x = 1$.

  $\cos^m x \sin^n x$ sine and cosine have even power, use 1/2-angle identites to simplify.

  $\cos mx \sin nx$ use appropriate trig identity to simplify.

  $\tan^m x \sec^n x$ m is odd, save $\sec x \tan x\, dx$, u-sub $u = \sec x$; use $\tan^2 x = \sec^2 x - 1$.

  $\tan^m x \sec^n x$ n is even, save $\sec^2 x\, dx$, u-sub $u = \tan x$; use $\sec^2 x = 1 - \tan^2 x$.

- rational $\longrightarrow$ partial fractions. Make sure degree of denominator is less than degree of numerator. If not, divide.

  distinct linear factors $\dfrac{4x}{(x-3)(x-4)} = \dfrac{A}{x-3} + \dfrac{B}{x-4}$

  repeated linear factors and distinct linear factors $\dfrac{3x-2}{(x-3)^2(x-4)} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{C}{x-4}$

  distinct quadratic factors and linear factors $\dfrac{1}{(x-3)^2(x^2+4x+5)} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{Cx+D}{x^2+4x+5}$

- product of two functions $\longrightarrow$ parts.

  one function becomes simpler when differentiated $\longrightarrow$ set u equal to this function

  Example $\int x \sin 4x\, dx,\quad u = x,\ dv = \sin 4x\, dx$

  neither function becomes simpler or more complex when differentiated $\longrightarrow$ use parts twice

  Example $\int e^{4x} \cos(3x)\, dx$

Step 4 Never, ever give up, if you can't think of how to start, try a substitution. Then maybe parts. Try *something*. If something doesn't work, how it doesn't work might give you a hint as to what will work.

Step 5 Most importantly, practice. With practice you will begin to recognize the patterns that tell you how to proceed.