Questions

Example $\int_0^\infty e^{-\pi x}\, dx$

Definition If a function $f(t)$ is continuous for $t \in [0, \infty)$, then its *Laplace transform* $F(s)$ is defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt.$$

The domain of F is the set consisting of all numbers s for which the integral converges.

Example Find the Laplace transform of $f(t) = e^t$.

Example Find the Laplace transform of $f(t) = 1$.

Example $\int_{-\infty}^\infty 1/(1 + x^2)\, dx$.

Solutions

Example $\int_0^\infty e^{-\pi x}\, dx$

Solution

$$\begin{aligned}
\int_0^\infty e^{-\pi x}\, dx &= \lim_{t\to\infty} \int_0^t e^{-\pi x}\, dx \text{ Substitution: } \begin{array}{ll} u = -\pi x, & x = 0 \longrightarrow u = 0 \\ du = -\pi\, dx, & x = t \longrightarrow u = -\pi t \end{array} \\
&= \lim_{t\to\infty} \frac{1}{(-\pi)} \int_0^{-\pi t} e^u\, du \\
&= -\frac{1}{\pi} \lim_{t\to\infty} e^u \Big|_0^{-\pi t} \\
&= -\frac{1}{\pi} \lim_{t\to\infty} \left(e^{-\pi t} - e^0\right) \\
&= -\frac{1}{\pi} \left(\lim_{t\to\infty} e^{-\pi t} - \lim_{t\to\infty} e^0\right) \\
&= -\frac{1}{\pi}(0 - 1) \\
&= \frac{1}{\pi}
\end{aligned}$$

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Example Find the Laplace transform of $f(t) = e^t$.

Solution

$$\begin{aligned}
F(s) &= \int_0^{\infty} f(t)e^{-st}\,dt \\
&= \int_0^{\infty} e^t e^{-st}\,dt \\
&= \int_0^{\infty} e^{(1-s)t}\,dt \\
&= \lim_{x\to\infty}\int_0^{x} e^{(1-s)t}\,dt \text{ Substitution: } \begin{array}{ll} u=(1-s)t, & t=0 \longrightarrow u=0 \\ du=(1-s)\,dt, & t=x \longrightarrow u=(1-s)x \end{array} \\
&= \lim_{x\to\infty}\frac{1}{(1-s)}\int_0^{(1-s)x} e^u\,du \\
&= \frac{1}{(1-s)}\lim_{x\to\infty} e^u\Big|_0^{(1-s)x} \\
&= \frac{1}{(1-s)}\lim_{x\to\infty}\left(e^{(1-s)x}-e^0\right) \\
&= \frac{1}{(1-s)}\left(\lim_{x\to\infty} e^{(1-s)x}-\lim_{x\to\infty} e^0\right) \\
&= \frac{1}{(1-s)}(0-1) \quad \text{only if } s>1 \text{, otherwise the first limit does not converge.} \\
&= \frac{-1}{(1-s)}, \quad s>1
\end{aligned}$$

The Laplace transform of $f(t) = e^t$ is $F(s) = -1/(1-s)$ where $s > 1$.

Example Find the Laplace transform of $f(t) = 1$.

Solution

$$\begin{aligned}
F(s) &= \int_0^{\infty} f(t)e^{-st}\,dt \\
&= \int_0^{\infty} (1)e^{-st}\,dt \\
&= \lim_{x\to\infty}\int_0^{x} e^{-st}\,dt \quad \text{Substitution: } \begin{array}{ll} u=-st, & t=0 \longrightarrow u=0 \\ du=-s\,dt, & t=x \longrightarrow u=-sx \end{array} \\
&= \lim_{x\to\infty}\frac{1}{(-s)}\int_0^{-sx} e^u\,du \\
&= \frac{1}{(-s)}\lim_{x\to\infty} e^u\Big|_0^{-sx} \\
&= -\frac{1}{s}\lim_{x\to\infty}\left(e^{-sx}-e^0\right) \\
&= -\frac{1}{s}\left(\lim_{x\to\infty} e^{-sx}-\lim_{x\to\infty} e^0\right) \\
&= -\frac{1}{s}(0-1) \quad \text{only if } s>0 \text{, otherwise the first limit does not converge.}
\end{aligned}$$

$$= \frac{1}{s}, \quad s > 0$$

The Laplace transform of $f(t) = 1$ is $F(s) = 1/s$ where $s > 0$.

Example $\int_{-\infty}^{\infty} 1/(1+x^2)\, dx$.

By symmetry, we can say that the integral is

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx &= 2\int_0^{\infty} \frac{1}{1+x^2}\, dx \\
&= 2\lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\, dx \\
&= 2\lim_{t\to\infty} \arctan x\big|_0^t \\
&= 2\lim_{t\to\infty} (\arctan t - \arctan(0)) \\
&= 2\lim_{t\to\infty} (\arctan t + 0) \\
&= 2\lim_{t\to\infty} \arctan t \\
&= 2\frac{\pi}{2} \\
&= \pi
\end{aligned}$$