Sequences: $\{a_n\}$

- The limit of a sequence $\lim_{n\to\infty} a_n$. If you use derivatives (L'Hospital's Rule) to evaluate limit, you must convert to a real function $f(x)$ where $f(n) = a_n$.
- Definitions of increasing, decreasing, monotonic.
- Definitions of bounded above, bounded below, bounded.
- To prove a sequence is decreasing, $b_{n+1} \leq b_n$, it is sometimes helpful to switch to a real valued function $f(x)$ and show $f'(x) < 0$.
- Monotonic Sequence Theorem: every bounded, monotonic sequence is convergent.
- The Test will not include mathematical induction.

Series: $\sum a_n$

- Sequence of partial sums $s_n = \sum_{i=1}^{n} a_i$. Note this means $a_n = s_n - s_{n-1}$.
- To find the sum of $\sum_{i=1}^{\infty} a_i$, we compute $\lim_{n\to\infty} s_n$.
- To sum a series, we need to get rid of the summation in s_n so we can take the limit (geometric, telescoping series).
- Types of convergence:

  $\sum a_n$ is convergent if $\sum a_n$ converges.
  $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.
  $\sum a_n$ is conditionally convergent if $\sum a_n$ converges and $\sum |a_n|$ diverges.

Two Very Good Test Series

The geometric series $\sum_{n=1}^{\infty} r^{n-1}$ converges to $\frac{1}{1-r}$ if $|r| < 1$. Divergent for $|r| \geq 1$. You should be able to prove this.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$. This can be shown with Integral Test.

The Seven Tests for Convergence or Divergence of a Series $\sum a_n$

Test for Divergence (pg 692) (if you notice $\lim_{n\to\infty} \neq 0$)

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Integral Test (pg 699) (if you notice $a_n = f(n)$ means $\int_c^\infty f(x)\,dx$ can be evaluated (check f is cont., pos., dec.))

Construct $f(x)$ from a_n so that $f(n) = a_n$. Then check that $f(x)$ is continuous, positive, and decreasing on the interval $[c, \infty)$. If this is all true, then you may use the integral test.

If $\int_c^\infty f(x)\,dx$ converges, then $\sum_{n=c}^{\infty} a_n$ converges.

If $\int_c^\infty f(x)\,dx$ diverges, then $\sum_{n=c}^{\infty} a_n$ diverges.

Comparison Test (pg 705) (if a_n looks like a geometric series or p-series, rational expression)

First, make sure that $\sum a_n$ and $\sum b_n$ have positive terms. If so, you can use the the comparison test.
If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent.
If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

Limit Comparison Test (pg 707) (if a_n looks like a geometric series or p-series, rational expression)

First, make sure that $\sum a_n$ and $\sum b_n$ have positive terms. If so, you can use the the limit comparison test. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$, $c > 0$ and finite, then either both series diverge or both converge.

Alternating Series Test (pg 710) (if series is an alternating series)

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n, \quad b_n > 0$, satisfies

1. $b_{n+1} \leq b_n$ for all n, and
2. $\lim_{n\to\infty} b_n = 0$,

then the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ is convergent.

The Ratio Test (pg 716) (if series has a factorial, or series contains $(r)^n$ $(r \neq -1)$)

$$\text{If } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \begin{cases} < 1 & \text{then } \sum a_n \text{ is absolutely convergent} \\ > 1 & \text{then } \sum a_n \text{ diverges} \\ = 1 & \text{the test fails} \end{cases}$$

The Root Test (pg 718) (if series is $a_n = (b_n)^n$)

$$\text{If } \lim_{n\to\infty}(|a_n|)^{1/n} = L \begin{cases} < 1 & \text{then } \sum a_n \text{ is absolutely convergent} \\ > 1 & \text{then } \sum a_n \text{ diverges} \\ = 1 & \text{the test fails} \end{cases}$$

Two Remainder Estimate Theorems

Remainder Estimate for the Integral Test (pg 701)

If the series $\sum a_n$ can be proven convergent using the integral test, then the remainder when using s_n to approximate the sum is the series is bounded by

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

If we add s_n to the inequalities, we get upper and lower bounds for the sum:

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx$$

Alternating Series Estimation Theorem (pg 712)

If $s = \sum(-1)^{n-1}b_n$, $b_n > 0$, is the sum of an alternating series that satisfies

1. $b_{n+1} \leq b_n$ for all n, and
2. $\lim_{n\to\infty} b_n = 0$,

then

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

How to choose a test when checking $\sum a_n$ for convergence

- There may be more than one test that will work.
- These are guidelines, not absolute rules.
- The tests don't find the sum of the series, they just tell you if the series is convergent/divergent.
- Practice is how to get good at this.

Advice For Comparison Tests: To decide which comparison test to use on $\sum a_n$:

- Examine a_n to see if as $n \to \infty$ the dominant terms look like a geometric or p-series.
- If so, use The Limit Comparison Test with comparison series b_n chosen from the dominant terms.
- If not, use The Comparison Test and build the comparison series b_n from the original series.

Is $\sum a_n$ a p-series or a geometric series?	Yes $\longrightarrow$	Use p-series result: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$. Diverges otherwise. Use geometric series result: $\displaystyle\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$ if $\lvert r\rvert < 1$. Diverges otherwise.
Is it obvious that $\lim\limits_{n\to\infty} a_n \neq 0$?	Yes $\longrightarrow$	Try the Test for Divergence.
Is $\sum a_n$ like a p-series or geometric series, and has positive terms?	Yes $\longrightarrow$	Try Limit Comparison Test or Comparison Test.
Is a_n a rational expression, or involves roots of polynomials?	Yes $\longrightarrow$	Try Limit Comparison Test or Comparison Test.
Is $\sum a_n$ an alternating series?	Yes $\longrightarrow$	Try Alternating Series Test.
Does a_n have factorials or constants raised to the n^{th} power?	Yes $\longrightarrow$	Try Ratio Test.
Does $a_n = (b_n)^n$?	Yes $\longrightarrow$	Try Root Test.
Is $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing and $\displaystyle\int_1^{\infty} f(x)\,dx$ can be easily evaluated?	Yes $\longrightarrow$	Try Integral Test.

On the Test, be prepared to (among other things, but these are most important):

- determine if a given series is absolutely convergent, conditionally convergent, or divergent,
- evaluate limits using l'Hospital's Rule or other techniques,
- demonstrate understanding of the concepts relating to sequences and series,
- prove the convergence result for the geometric series, in general and for specific geometric series,
- sum a telescoping series using partial fractions,
- prove the test for divergence,
- prove the remainder estimate for the integral test,
- do integrals (needed for the integral test, so parts and u-sub are most likely to occur).

Evaluating Limits

If $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is an indeterminant quotient ($\frac{\infty}{\infty}$ or $\frac{0}{0}$) , then by l'Hospital's Rule:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{d}{dx}[f(x)]}{\frac{d}{dx}[g(x)]}.$$

Divide by highest power of n for rational expressions:

$$\lim_{x\to\infty} \frac{14n^2+17}{2n^2-3n+2} = \lim_{x\to\infty} \frac{14+\frac{17}{n^2}}{2-\frac{3}{n}+\frac{2}{n^2}} = \frac{14+0}{2-0+0} = 7$$

Limits for indeterminant powers can usually be avoided by switching to a Ratio Test from a Root Test, but they can be evaluated using logarithms:

$$\begin{aligned}
&\lim_{x\to\infty} x^{1/x} \to \infty^0 \text{ (indeterminant power)}\\
&y = x^{1/x}\\
&\ln y = \frac{\ln x}{x}\\
&\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \longrightarrow \frac{\infty}{\infty}\\
&\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{x} = 0 \text{ (l'Hospital's Rule)}\\
&\lim_{x\to\infty} e^{\ln y} = e^0\\
&\lim_{x\to\infty} y = 1
\end{aligned}$$

Factorials

The factorial is defined as $n! = 1\cdot 2\cdot\dots\cdot(n-1)(n)$. Note that $0! = 1$ and $1! = 1$.

To simplify factorials, expand and cancel:

$$\frac{(4n)!}{(4n+2)!} = \frac{\cancel{(4n)!}}{\cancel{(4n)!}(4n+1)(4n+2)} = \frac{1}{(4n+1)(4n+2)}$$

Integration

$$\int_1^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}dx = -\frac{1}{2}\lim_{t\to\infty}\int_{-1}^{-t^2} e^u du = -\frac{1}{2}\lim_{t\to\infty}\left(e^u\right)_{-1}^{-t^2} = -\frac{1}{2}\lim_{t\to\infty}(e^{-t^2}-e^{-1}) = \frac{1}{2e}$$

sub $u = -x^2$, so $du = -2x\,dx$, and when $x = 1, u = -1$ and $x = t, u = -t^2$.