1102 Calculus II Chapter 11 Sections 11.8–11.11 Review

Definitions

A power series is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

- this is a power series about a,
- the c_n are the coefficients of the power series,
- the series may converge or diverge for each value of x.

The Taylor series of f about $x = a$ is $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R.$

- The radius of convergence of the Taylor series is R. It is often found using the Ratio Test.
- The interval of convergence of the Taylor series is the interval on which it converges. The endpoints of the interval, $x = a \pm R$, have to be checked separately using different tests for convergence since this is where the Ratio Test fails (comparison tests are often good ones to try).

The MacLaurin series of f is the Taylor series about $x = 0$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n, \quad |x| < R.$

The Taylor polynomials of f about $x = a$ are $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$

The Remainder is $R_n(x) = f(x) - T_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i.$

Finding Power series for $f(x)$

We have three ways of getting power series for a function $f(x)$.

1. If the function $f(x)$ can be manipulated to look like $\frac{1}{1-y}$, then we can use the geometric series result:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n, \qquad |y| < 1.$$

   The radius of convergence is found from the $|y| < 1$ condition.

2. If the function $f(x)$ can be manipulated to look like $(1+y)^k$, then we can use the binomial series result:

$$(1+y)^k = \sum_{n=0}^{\infty} \binom{k}{n} y^n, \quad |y| < 1. \quad \text{(works for all } k \in \mathbb{R}\text{)}$$

$$(1+y)^k = \sum_{n=0}^{\infty} \frac{k!}{(k-n)!n!} y^n, \quad |y| < 1. \quad \text{(does not work for } k \text{ a positive integer since series is not infinite)}$$

The radius of convergence is found from the $|y| < 1$ condition.

3. If neither of the above are possible, the general Taylor series method must be used where you create the table with the derivatives $f^{(n)}(x)$ and $f^{(n)}(a)$ and you try to find a pattern in the derivatives. The radius of convergence is found using the Ratio Test.

Common Taylor Series

You should know the Taylor series about $x = 0$ for common functions (see page 743 in text):

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \text{ geometric series} \\
&= 1 + x + x^2 + x^3 + \cdots \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty \\
&= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1 \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} y^n, \quad |y| < 1 \\
&= 1 + kx + \frac{k(k-1)x^2}{2} + \frac{k(k-1)(k-2)x^3}{6} + \frac{k(k-1)(k-2)(k-3)x^4}{24} + \cdots
\end{aligned}$$

Concepts

- The Taylor polynomial $T_n(x)$ of a power series about $x = a$ approximates the function best near $x = a$.
- Manipulating functions to create new power series from the previously computed Taylor series.
  - Substitution

$$\begin{aligned}
\frac{2}{1+3x} &= 2\left(\frac{1}{1-(-3x)}\right) \\
&= 2\sum_{n=0}^{\infty}(-3x)^n, \ \ |-3x| < 1 \\
&= 2\sum_{n=0}^{\infty}(-1)^n 3^n x^n, \ \ |x| < \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
xe^{-x^3/2} &= x\sum_{n=0}^{\infty}\frac{(-x^3/2)^n}{n!}, \ \ |(-x^3/2)| < \infty \\
&= x\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n}}{2^n n!}, \ \ |x| < \infty \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+1}}{2^n n!}, \ \ |x| < \infty
\end{aligned}$$

  - Differentiation

$$\begin{aligned}
\frac{d}{dx}\ln(3+2x) &= \frac{2}{3+2x} \\
&= \frac{2}{3}\left(\frac{1}{1+2x/3}\right) \\
&= \frac{2}{3}\left(\frac{1}{1-(-2x/3)}\right) \\
&= \frac{2}{3}\sum_{n=0}^{\infty}\left(-\frac{2x}{3}\right)^n, \ \ |-\frac{2x}{3}| < 1 \\
&= \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{3^n}x^n, \ \ |x| < \frac{3}{2} \\
\ln(3+2x) &= \int \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{3^n}x^n\,dx, \ \ |x| < \frac{3}{2} \\
&= \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{3^n}\int x^n\,dx, \ \ |x| < \frac{3}{2} \\
&= \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{3^n}\frac{x^{n+1}}{n+1} + K, \ \ |x| < \frac{3}{2} \\
\ln(3) &= 0 + K \\
\ln(3+2x) &= \frac{2}{3}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{3^n}\frac{x^{n+1}}{n+1} + \ln 3, \ \ |x| < \frac{3}{2}
\end{aligned}$$

- Estimation of error:
  - graphically,
  - if series alternates, use alternating series estimation theorem.

Mathematica

- Plotting $f(x)$ and $T_n(x) = \sum_{i=0}^{n} c_i(x-a)^i$ when you have worked out the pattern and know c_n

```
f[x_] = Exp[x]
a = 0
c[n_] = 1/n!
T[n_, x_] := Sum[c[i](x - a)^i, {i, 0, n}]

Plot[{f[x], T[7, x]}, {x, -10, 10}, PlotRange -> {{-5, 5}, {-3, 10}}]
```

- Finding the first few terms in power series of $f(x)$ about $x = a$ (will not give you the pattern for the power series)

$$f(x) \sim T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

```
f[x_] = Cos[x]^4/(1 - x^2)
a = Pi
T[n_, x_] := Sum[Derivative[i][f][a]/i!(x - a)^i, {i, 0, n}]
T[3, x]
```

- Built-in command to find Taylor Polynomials of f about $x = a$ to order n

```
f[x_] = Cos[x]^4/(1 - x^2)
Series[f[x], {x, Pi, 3}]
Series[f[x], {x, Pi, 3}] // Normal
```

- Built-in command to find general coefficient in power series, $\frac{f^{(n)}(a)}{n!}$. This replaces what we do by hand when we find the derivatives, which you may find useful in the future but I will want you to work this out by hand and looking for the pattern. You can use this to check your answer, although it is often hard to interpret the output for more complicated cases.

```
SeriesCoefficient[Exp[x], {x, 0, n}]
```

Example Find the fifth degree Taylor Polynomial approximation to $f(x) = x\cos(\beta\pi x)$ about $x = 0$. For $\beta = 1.4$, plot $T_5(x)$ and $f(x)$ for $-1 < x < 1$ and $-1 < y < 1$. Clearly label which function is which in your sketch.

Solution Since we are <u>not</u> looking for a pattern, it may be easier to work out the derivatives by hand outside of a table. We can use *Mathematica* for that.

```
Clear[beta]
f[x_] = x*Cos[beta*Pi*x];
a = 0;
T[n_, x_] := Sum[Derivative[i][f][a]/i! (x - a)^i, {i, 0, n}]
T[5, x]
beta = 1.4
Plot[{f[x], T[5, x]}, {x, -1, 1}, PlotLegends -> {"f(x)", "T5(x)"}]
```

We find that $T_5(x) = x - \frac{1}{2}\beta^2\pi^2x^3 + \frac{1}{24}\beta^4\pi^4x^5$.