

1102 Calculus II Chapter 9 Review

Topics

- Differential equations

Solutions to differential equations are families of curves. If we have some condition that the solution must satisfy, then we pick a curve from the family of curves.

A differential equation with initial condition is called an initial value problem.

The equilibrium solutions are constant valued solutions, $y(t) = \text{constant}$.

- Direction fields

A direction field is a graphical way of visualizing the solution to a differential equation. It will show the family of curves which solve the differential equation.

Be prepared to match direction fields to differential equations, and justify your picks. In other words, be able to construct at least parts of a direction field, and be able to interpret a direction field.

- Euler's method

Euler's method is a numerical method which is used to determine the solution to initial value problems,

$$y' = f(x, y), \quad y(x_0) = y_0.$$

It can be motivated graphically or analytically.

The iteration equation is typically written in the following form:

$$\begin{aligned} x_0 \text{ and } y_0 \text{ are given. You might choose an } h \text{ or it will be given also.} \\ x_n &= x_{n-1} + h \\ y_n &= hf(x_{n-1}, y_{n-1}) + y_{n-1} \end{aligned}$$

and we let $n = 1, 2, 3, \dots$ to get the solution. The initial point (x_0, y_0) and step size h are given.

What we get out is a set of points (x_n, y_n) , which approximate the solution to the differential equation.

- Separable equations

The method of solution for separable equations:

1. put all the y 's on one side, all the x 's on the other (including differentials),
2. integrate,
3. simplify.

Note that it is important to treat constant of integration properly.

- Orthogonal trajectories

Two curves are orthogonal at a point if their derivatives are negative reciprocals of each other at that point.

An orthogonal trajectory of a family of curves is a curve that intersects each curve of the family of curves orthogonally.

To find the orthogonal trajectories of a given family of curves, we need to do a few things:

1. Construct the differential equation the given family of curves satisfies.
2. Use the fact that orthogonal trajectories have derivatives which are negative reciprocals to determine the differential equation the orthogonal trajectories must satisfy.
3. Solve this differential equation to get the orthogonal trajectories.

- Exponential growth or decay from data

This is the model that develops when you assume the population grows at a rate proportional to the size of the population, which gives us the differential equation

$$\frac{dy}{dt} = ky$$

where k is the population growth constant, and y is the population at time t .

The solution to the differential equation is

$$y(t) = Ae^{kt},$$

and use two data points to determine the two constants A and k .

We always want to use data points close to the values we wish to use the model to predict.

If we are also given an initial condition $y(t_0) = y_0$, then the solution to the initial value problem

$$\frac{dy}{dt} = ky, \quad y(t_0) = y_0$$

is

$$y(t) = y_0 e^{k(t-t_0)}.$$

We use another data point to determine k , or k may be given to us.

If $k > 0$, we have exponential growth.

If $k < 0$, we don't have growth, we have exponential decay (used to model radioactive decay).

Half-life is the time it takes half an amount of substance to disappear.

- Other Population Growth Models

$$\text{Doomsday Equation: } \frac{dy}{dt} = ky^{1+w}, \quad y(t_0) = y_0, \quad w > 0 \Rightarrow y(t) = \left(\frac{1}{-kw(t-t_0) + y_0^{-w}} \right)^{1/w}$$

$$\text{Logistic Equation: } \frac{dy}{dt} = ky \left(1 - \frac{y}{K} \right), \quad y(t_0) = y_0, \quad K > 0 \Rightarrow y(t) = \frac{y_0 K}{y_0 + (K - y_0) e^{-k(t-t_0)}}$$

$$\text{Gompertz Equation: } \frac{dy}{dt} = ky \ln \left(\frac{K}{y} \right), \quad y(t_0) = y_0, \quad K > 0 \Rightarrow y(t) = K \left(\frac{y_0}{K} \right)^{e^{-k(t-t_0)}}$$

The Doomsday solution becomes infinite at a finite time.

K is the carrying capacity (maximum population the resources can support).

- Mathematica (not on this test)

Example: $\frac{dy}{dx} = 6x^2 - 3x^2y, \quad y(1) = 3.$

Euler's Method:

```
Clear[x, y, f, h]
x[0] = 1
y[0] = 3
h = 0.01
f[x_, y_] = 6x^2 - 3x^2y
x[n_] := x[n] = x[n - 1] + h
y[n_] := y[n] = y[n - 1] + h f[x[n - 1], y[n - 1]]
data = Table[{x[n], y[n]}, {n, 0, 200}];
TableForm[%]
```

Plotting points from Euler's Method:

```
dataplot = ListPlot[data, PlotStyle -> {Red, PointSize[0.02]}, AxesLabel -> {"x", "y"}]
```

Creating Direction Fields:

```
<< Graphics`PlotField`
```

```
field = PlotVectorField[{1, 6x^2 - 3x^2y}, {x, 0, 3}, {y, 1, 4},
  AspectRatio -> 1, Frame -> True, ImageSize -> {500, Automatic}, FrameLabel -> {"x", "y"}]
```

```
Show[field, dataplot]
```

Example Solve the differential equation $\frac{dy}{dt} = r - ay$, where r and a are constants.

$$\begin{aligned} \frac{dy}{dt} &= r - ay \\ \frac{dy}{r - ay} &= dt \quad (\text{separate}) \\ \int \frac{dy}{r - ay} &= \int dt \quad (\text{integrate}) \\ \int \frac{dy}{r - ay} &= t + c_1 \quad \text{Substitute: } u = r - ay, \quad du = -ady \\ \frac{1}{-a} \int \frac{du}{u} &= t + c_1 \\ \frac{1}{-a} \ln |u| + c_2 &= t + c_1 \\ -\frac{1}{a} \ln |r - ay| &= t + c \quad (c = c_1 - c_2) \end{aligned}$$

This is an implicit solution to the differential equation. To get the explicit solution we need to do some algebra.

$$\begin{aligned} \ln |r - ay| &= -at - ac \\ |r - ay| &= \exp(-at - ac) \\ r - ay &= \pm \exp(-ac) \exp(-at) \\ r - ay &= k \exp(-at), \quad \text{where } k = \pm \exp(-ac) \\ y &= \frac{r}{a} + \frac{k}{a} \exp(-at) \\ y &= \frac{r}{a} + B e^{-at}, \quad \text{where } B = k/a. \end{aligned}$$

B is the constant of integration. The constants r and a were constants given in the initial problem.

Example After four days a sample of radon-222 decayed to 50% of its original amount. How long would it take the sample to decay to 1% of its original amount?

We don't know the initial mass, but we can assume it is m_0 . We will assume this is at time $t_0 = 0$ (t will be measured in days). This gives us the initial point $(t_0, m_0) = (0, m_0)$.

If we let $y(t)$ be the mass at any time t , then the model for radioactive decay is

$$\begin{aligned} y(t) &= y_0 e^{k(t-t_0)} \quad (\text{the model}) \\ y(t) &= m_0 e^{kt} \quad (\text{using the point } (t_0, y_0) = (0, m_0)) \end{aligned}$$

Now we need to determine k . After 4 days the mass is $m_0/2$. The second point we need is $(4, m_0/2)$.

$$\begin{aligned} y(4) &= m_0/2 = m_0 e^{4k} \quad (\text{using the point } (4, m_0/2)) \\ k &= \frac{1}{4} \ln\left(\frac{1}{2}\right) \quad (\text{solve for } k) \\ y(t) &= m_0 e^{\frac{1}{4} \ln(\frac{1}{2})t} \quad (\text{the model}) \end{aligned}$$

The time to decay to 1% of its mass will be found by solving the following equation for t

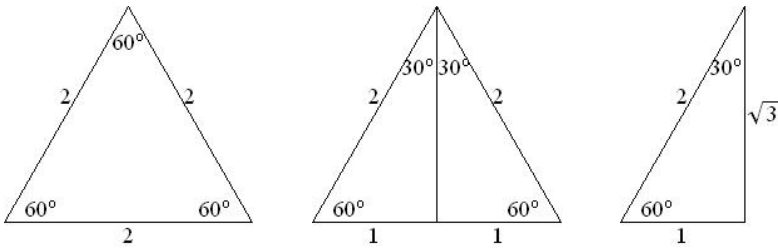
$$y(t) = m_0/100 = m_0 e^{\frac{1}{4} \ln(\frac{1}{2})t} \longrightarrow t = \frac{4 \ln(1/100)}{\ln(1/2)} = \frac{4 \ln(100)}{\ln(2)} \text{ days.}$$

Example Solve the initial value problem $\frac{du}{dr} = (1 + \sqrt{r}) \sec^3 u$, $u(0) = \pi/3$.

$$\begin{aligned} \cos^3 u \, du &= (1 + \sqrt{r}) \, dr \\ \int \cos^3 u \, du &= \int (1 + \sqrt{r}) \, dr \\ \int \cos^2 u \cos u \, du &= r + \frac{2}{3} r^{3/2} + c \\ \int (1 - \sin^2 u) \cos u \, du &= r + \frac{2}{3} r^{3/2} + c \\ &\quad \text{Substitution: } w = \sin u, \, dw = \cos u \, du \\ \int (1 - w^2) \, dw &= r + \frac{2}{3} r^{3/2} + c \\ w - \frac{1}{3} w^3 &= r + \frac{2}{3} r^{3/2} + c \\ \sin u - \frac{1}{3} \sin^3 u &= r + \frac{2}{3} r^{3/2} + c \end{aligned}$$

Now, use the initial condition to determine c :

$$\sin(\pi/3) - \frac{1}{3} \sin^3(\pi/3) = 0 + \frac{2}{3} 0^{3/2} + c$$



$$\Rightarrow \sin(\pi/3) = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

$$\frac{\sqrt{3}}{2} - \frac{1}{3} \left(\frac{\sqrt{3}}{2} \right)^3 = c = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8} = \frac{3\sqrt{3}}{8}$$

The (implicit) solution to the initial value problem is

$$\sin u - \frac{1}{3} \sin^3 u = r + \frac{2}{3} r^{3/2} + \frac{3\sqrt{3}}{8}$$

Getting an explicit solution is possible, but tricky since we have a cubic trig equation in u .

Example Find the orthogonal trajectories for the family of curves $x^3 - \sec y = k$.

We begin by constructing the differential equation the family of curves satisfies:

$$\begin{aligned} \frac{d}{dx} [x^3 - \sec y] &= k \\ 3x^2 - \sec y \tan y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{3x^2}{\sec y \tan y} \end{aligned}$$

Now, we take negative reciprocal of the derivative to get the DE the orthogonal trajectories will satisfy:

$$\begin{aligned} -\frac{dx}{dy} &= \frac{3x^2}{\sec y \tan y} \\ -\frac{dx}{3x^2} &= \frac{1}{\sec y \tan y} dy \\ -\frac{1}{3} \int x^{-2} dx &= \int \frac{\cos y \cos y}{\sin y} dy \\ \frac{1}{3} x^{-1} &= \int \frac{\cos^2 y}{\sin y} dy \\ \frac{1}{3x} &= \int \frac{1 - \sin^2 y}{\sin y} dy \\ \frac{1}{3x} &= \int (\csc y - \sin y) dy \\ \frac{1}{3x} &= \ln(\csc y - \cot y) + \cos y + c \end{aligned}$$

This is an implicit expression for the orthogonal trajectories to the original family of curves.