Euler's Formula (For Your Enjoyment–Will Not Be On Tests)

Using Taylor series we can get an amazing result, known as Euler's formula, which relates exponential functions, sines and cosines.

First, we need to introduce the complex number i, which is defined by $i^2 = -1$. This means we can write things like $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$.

Note that this means:

$$\begin{aligned}
&i^0 = 1,\\
&i^1 = i,\\
&i^2 = -1,\\
&i^3 = i \cdot i^2 = -i,\\
&i^4 = i^2 \cdot i^2 = 1,\\
&i^5 = i^4 \cdot i = i,\\
&i^6 = i^4 \cdot i^2 = -1,\\
&i^7 = i^6 \cdot i = -i,
\end{aligned}$$

Here's the awesomeness, in which we only are taking a few things on faith at this point (like *What the heck is a function of a complex number?* If you want to know more, you will want to study complex analysis!):

Taylor Series for exponential:

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}. \quad \text{converges everywhere}$$

Evaluate at $u = ix$:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!}. \quad \text{converges everywhere}$$

Expand out the infinite sum:

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots.$$

Simplify the powers of i using the results from above:

$$e^{ix} = 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots.$$

Collect the terms without an i, and those with an i:

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right).$$

Use summation notation to fold up the two infinite sums:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right).$$

Identify the sums as cosine and sine:

$$e^{ix} = \cos x + i\sin x.$$

This is Euler's Formula: $\boxed{e^{ix} = \cos x + i\sin x}$

Mathematica verifies this is a valid formula.

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ComplexExpand[Exp[I*x]]
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What is the Value of $\ln(-1)$?

The logarithm of a negative is not defined... well, that's not really true–what is true is *the logarithm of a negative is not a real number.*

Let $x = \pi$:

$$\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{i\pi} &= \cos\pi + i\sin\pi \\ e^{i\pi} &= -1 + i(0) \\ e^{i\pi} &= -1 \\ \ln e^{i\pi} &= \ln(-1) \\ i\pi &= \ln(-1) \end{aligned}$$

So it looks like $\ln(-1) = i\pi$. Amazing. It is. Ask *Mathematica*.

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Log[-1]
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The formula (known as *Euler's identity*) $e^{i\pi}+1=0$ is supercalifragilistic since it relates four fundamental numbers: $0, 1, e, \pi$.

What are the Trig Angle Addition Identities?

Do you even remember those trig identities, let alone how to prove them? This is an easy way to remember them, and even more important understand what they actually mean!

Let $x = u + v$:

$$\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{i(u+v)} &= \cos(u+v) + i\sin(u+v) \\ e^{i(u+v)} &= e^{iu}e^{iv} \\ &= (\cos u + i\sin u)(\cos v + i\sin v) \\ &= \cos u\cos v + i^2\sin u\sin v + i\sin u\cos v + i\cos u\sin v \\ &= \cos u\cos v - \sin u\sin v + i(\sin u\cos v + \cos u\sin v) \end{aligned}$$

Comparing the two expressions for $e^{i(u+v)}$, we see that we must have

$$\begin{aligned} \cos(u+v) &= \cos u\cos v - \sin u\sin v \\ \sin(u+v) &= \sin u\cos v + \cos u\sin v \end{aligned}$$

Define Sine and Cosine in Terms of Exponential

Hey, you want to replace some sines and cosines with exponentials? Here is how to do that!

$$\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{-ix} &= \cos(-x) + i\sin(-x) = \cos x - i\sin x \end{aligned}$$

Add the equations to get: $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$.

Subtract the equations to get: $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$.

Hyperbolic Cosine Function

You have seen the function $\cosh x$ sprinkled throughout the text. What is it?

Let $x \to ix$ in the equation we just derived:

$$\begin{aligned} \cos x &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \\ \cos(ix) &= \frac{1}{2}\left(e^{i(ix)} + e^{-i(ix)}\right) = \frac{1}{2}\left(e^{-x} + e^{x}\right) = \cosh x \end{aligned}$$

The hyperbolic cosine is the cosine with an imaginary argument, $\cosh x = \cos(ix)$.

You can do something similar to get the relation $\sinh x = -i\sin(ix) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$.

From this, we can see calculus results like:

$$\begin{aligned} \frac{d}{dx}[\cosh x] &= \frac{d}{dx}\left[\frac{1}{2}\left(e^{-x} + e^{x}\right)\right] \\ &= \frac{1}{2}\left(-e^{-x} + e^{x}\right) \\ &= \sinh x \end{aligned}$$

Notice there is no minus sign! In regular trig, $\frac{d}{dx}[\cos x] = -\sin x$.

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D[Cosh[x],x]
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The hyperbolic functions come up often in physics and engineering. They are based on a hyperbola rather than a circle like our regular trig functions. There are hyperbolic identities like $\cosh^2 x - \sinh^2 x = 1$.

What is i^i?

This may not be the most useful result, but it is one of my favourites!

Let $x = \pi/2$ in Euler's formula:

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
e^{i\pi/2} &= \cos(\pi/2) + i \sin(\pi/2) \\
e^{i\pi/2} &= i \\
(e^{i\pi/2})^i &= i^i \\
e^{i^2\pi/2} &= i^i \\
e^{-\pi/2} &= i^i
\end{aligned}$$

So it looks like $i^i = e^{-\pi/2}$. Amazing. It is. Ask *Mathematica*.

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ComplexExpand[I^I]
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This means $e^{-\pi/2} - \sqrt{-1}^{\sqrt{-1}} = 0$. I think I like this better than Euler's identity!

What is $\sqrt[i]{i}$?

This looks terrifying at first, the i^{th} root of i, or $\sqrt[\sqrt{-1}]{\sqrt{-1}}$. Is it some sort of Zombie Apocalypse? But figuring out what this simplifies to starts just like i^i.

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
e^{i\pi/2} &= \cos(\pi/2) + i \sin(\pi/2) \\
e^{i\pi/2} &= i \\
(e^{i\pi/2})^{1/i} &= \sqrt[i]{i} \qquad \text{use } \frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = \frac{i}{i^2} = -i \\
(e^{i\pi/2})^{-i} &= \sqrt[i]{i} \\
e^{-i^2\pi/2} &= \sqrt[i]{i} \\
e^{\pi/2} &= \sqrt[i]{i}
\end{aligned}$$

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ComplexExpand[I^(1/I)]
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Further Reading

If you want to learn more about *hyperbolic functions*, check out
http://en.wikipedia.org/wiki/Hyperbolic_function#Useful_relations.

There is a wonderful reference to help you understand Euler's formula in more detail if you find this interesting.
http://betterexplained.com/articles/intuitive-understanding-of-eulers-formula/

If you want to learn more about the *Zombie Apocalypse*, check out
http://en.wikipedia.org/wiki/Zombie#Zombie_apocalypse