Power Series

Now, we have three ways of getting power series for a function $f(x)$.

1. If the function $f(x)$ can be manipulated to look like $\frac{1}{1-y}$, then we can use the geometric series result:
$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n, \qquad |y| < 1.$$
The radius of convergence is found from the $|y| < 1$ condition.
2. If the function $f(x)$ can be manipulated to look like $(1+y)^k$, then we can use the binomial series result:
$$(1+y)^k = \sum_{n=0}^{\infty} \binom{k}{n} y^n = \sum_{n=0}^{\infty} \frac{k!}{(k-n)!n!} y^n, \qquad |y| < 1.$$
The radius of convergence is found from the $|y| < 1$ condition.
3. If neither of the above are possible, the general Taylor series method must be used where you create the table with the derivatives $f^{(n)}(x)$ and $f^{(n)}(a)$ and you try to find a pattern in the derivatives. The radius of convergence is found using the Ratio Test.

The Factorial

Let's look at an example that leads to some interesting ideas about the factorial function.

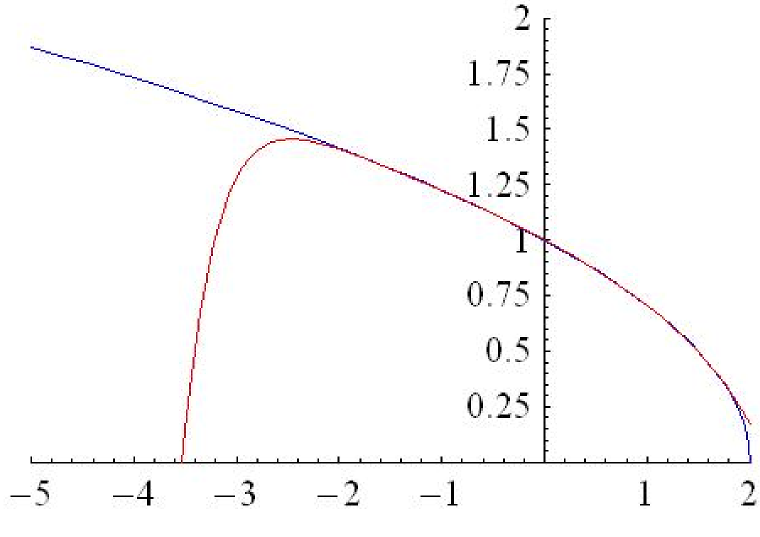
Example Find a power series for $\sqrt{1-x/2}$ about $x = 0$.

This look like a binomial series, so we can use the following:

$$\begin{aligned}
\sqrt{1-x/2} &= \sqrt{1+\left(-\frac{x}{2}\right)} \\
&= \sum_{n=0}^{\infty} \frac{(1/2)!}{(1/2-n)!n!} \left(-\frac{x}{2}\right)^n, \qquad \left|-\frac{x}{2}\right| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)!}{2^n(1/2-n)!n!} x^n, \qquad |x| < 2
\end{aligned}$$

If we go ahead and plot the Taylor polynomial $T_{10}(x)$ against the original $f(x)$, we see that we have the correct power series.

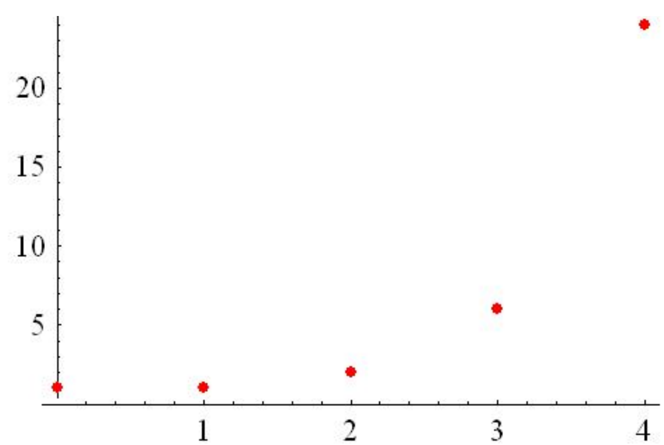
```
f[x_] = Sqrt[1 - x/2]

T[n_, x_] := Sum[(-1)^i(1/2)!/2^i/(1/2 - i)!/i!*x^i, {i, 0, n}]

Series[f[x], {x, 0, 4}]
T[4, x]

Plot[{f[x], T[10, x]}, {x, -5, 2}, PlotStyle -> {Blue, Red},
  PlotRange -> {{-5, 2}, {0, 2}}]
```



This all looks great (and it is), until we look a bit closer and see the (1/2)!. What does that mean? Let's investigate.

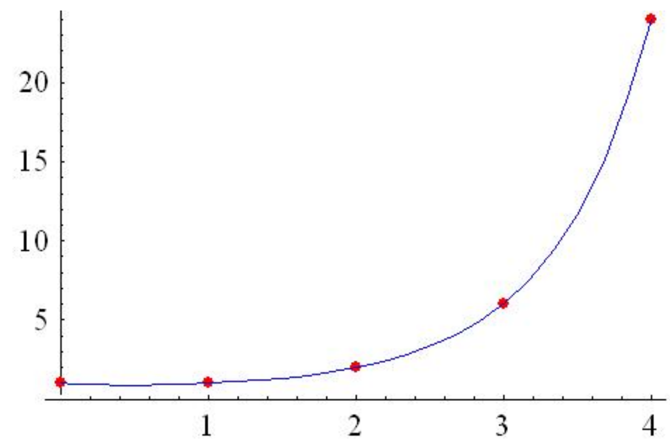
The factorial for integers is defined as $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)n$, with $0! = 1$. Let's plot these points:

```
list1 = Table[{n, n!}, {n, 0, 4}]
plot1 = ListPlot[list1, PlotStyle -> {Red, PointSize[0.02]}, AxesOrigin -> {0, 0}]
```



Could we draw a smooth line through these points, which would then define the factorial function at the real numbers instead of just integers? It sure looks like we could. Let's see if *Mathematica* will do it:

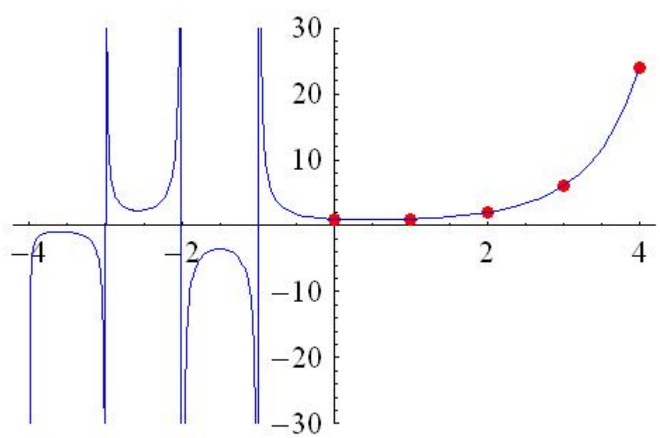
```
Plot[x!, {x, 0, 4}, PlotStyle -> {Blue}, PlotRange -> {{-5, 2}, {0, 4}}]
Show[plot1, %, PlotRange -> {All, All}]
```



Amazing–the factorial <u>is</u> defined for non-integers! *Mathematica* can tell us that $(1/2)! = \sqrt{\pi}/2$. This π is canceled out by the $(1/2 - i)!$, so our Taylor series earlier did not contain any π.

What about negative numbers? Let's try it:

```
Plot[x!, {x, -4, 4}, PlotStyle -> {Blue}, PlotRange -> {{-5, 2}, {0, 4}}]
Show[plot1, %, PlotRange -> {All, {-30, 30}}]
```



Wow! Note those are vertical asymptotes at $x = -1, -2, -3, \ldots$. For this reason, we should use the binomial coefficient $\binom{k}{n}$ instead of the factorials when working with binomial series where k is a negative integer.

In fact, the factorial at integers is just a special value of another function, the gamma function $\Gamma(x)$, that is defined as follows:

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1} e^{-t}\, dt \\ \Gamma(n) &= (n-1)! \qquad \text{where } n \text{ is an integer}\end{aligned}$$

Understanding the gamma function is something you will do if you take some more advanced math classes.