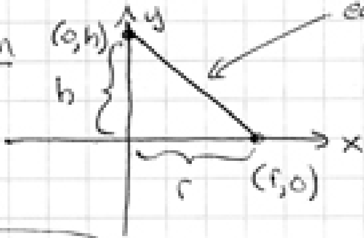
Ex Verify the surface area of a right circular cone of height h and radius r is $\pi r\sqrt{r^2+h^2}$

solution

sketch

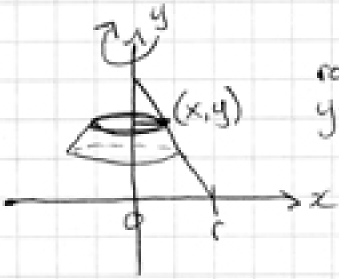


equation of line:

$$\frac{y-y_0}{x-x_0} = \frac{y_1-y_0}{x_1-x_0}$$

OR slope $= -\frac{h}{r}$

y-intercept $= h$



rotate about y-axis.

Equation is $y = mx+b$

$$\boxed{y = -\frac{hx}{r} + h}$$

Circumference of circle $= 2\pi x$

surface area of part of cone $\approx 2\pi x\, ds$

Surface area of cone $= \int 2\pi x\, ds$

$$ds = \sqrt{(dx)^2+(dy)^2}$$

We have a choice - let's choose to integrate over x and see what happens.

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{1+\left(-h/r\right)^2}\,dx$$

$$= \sqrt{1+h^2/r^2}\,dx$$

$$= \frac{\sqrt{r^2+h^2}}{r}\,dx$$

so Surface Area of cone $= \int_0^r 2\pi x \frac{\sqrt{r^2+h^2}}{r}\,dx$

$$= \frac{2\pi\sqrt{r^2+h^2}}{r}\int_0^r x\,dx$$

$$= \frac{2\pi}{r}\sqrt{r^2+h^2}\left.\frac{x^2}{2}\right|_0^r$$

$$= \pi r\sqrt{r^2+h^2}$$

Here is what would have happened if we decided to integrate with respect to y when we started working with ds:

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy$$

$$y = -\frac{h}{r}x + h \quad \rightarrow \quad y - h = -\frac{h}{r}x$$

$$x = -\frac{r}{h}(y-h)$$

$$\frac{dx}{dy} = -\frac{r}{h}$$

$$ds = \sqrt{\left(-r/h\right)^2 + 1}\; dy$$

$$= \sqrt{\frac{r^2}{h^2} + 1}\; dy$$

$$= \frac{\sqrt{r^2+h^2}}{h}\; dy$$

Surface Area of cone $$= \int_0^h 2\pi \left(-\frac{r}{h}(y-h)\right) \frac{\sqrt{r^2+h^2}}{h}\, dy$$

$$= -\frac{2\pi r\sqrt{r^2+h^2}}{h^2} \int_0^h (y-h)\, dy$$

$$= -\frac{2\pi r}{h^2}\sqrt{r^2+h^2}\left(\frac{y^2}{2} - hy\right)\Big|_0^h$$

$$= -\frac{2\pi r}{h^2}\sqrt{r^2+h^2}\left(\frac{h^2}{2} - h^2\right)$$

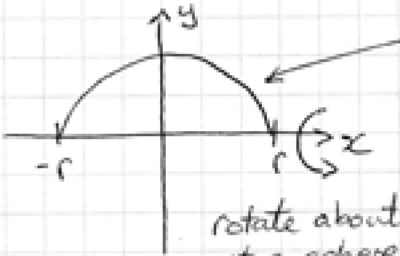
$$= -\frac{2\pi r}{h^2}\sqrt{r^2+h^2}\left(-\frac{h^2}{2}\right)$$

$$= \pi r\sqrt{r^2+h^2}$$

Ex Find the surface area of a sphere of radius r using calculus.

Solution: we will need a region (curve, actually) in the xy-plane which we can rotate about an axis to generate a sphere of radius r.

sketch:



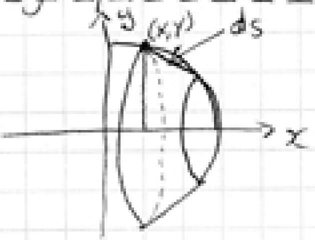
Explicit: $y = \sqrt{r^2 - x^2}$, $-r < x < r$.

OR

~~Implicit~~ Parametric: $x = r\cos t$, $y = r\sin t$, $0 < t < \pi$.

rotate about x-axis to get a sphere of radius r.

Using Explicit function:



$y = \sqrt{r^2 - x^2}$

Circumference of circle $= 2\pi y$

Surface Area of part of cone $\sim 2\pi y\, ds$

Surface area of sphere $= 2\int_0^r 2\pi y\, ds$

$$= 2\int_0^r 2\pi \sqrt{r^2 - x^2}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}}$$

$$\text{Surface Area} = 4\pi\int_0^r \sqrt{r^2 - x^2}\,\frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 4\pi r\int_0^r dx = 4\pi r\, x\Big|_0^r = 4\pi r^2$$

Using Parametric function:

The sketch is the same as for explicit function, and we get

$$\begin{array}{l}\text{Surface}\\\text{area of}\\\text{sphere}\end{array} = \int_0^\pi 2\pi y\, ds$$

$$= \int_0^\pi 2\pi\, r\sin t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -r\sin t \qquad \frac{dy}{dt} = r\cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$

$$= r\sqrt{\sin^2 t + \cos^2 t}$$

$$= r$$

$$\text{Surface Area} = 2\pi r\int_0^\pi r\sin t\, dt$$

$$= 2\pi r^2(-\cos t)\Big|_0^\pi$$

$$= 2\pi r^2(-\cos\pi + \cos 0)$$

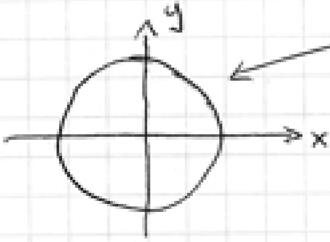
$$= 2\pi r^2(-(-1) + 1)$$

$$= 4\pi r^2$$

$\boxed{\text{Ex}}$ Verify the circumference of a circle of radius r is $2\pi r$ using calculus.

Solution We need a sketch of a circle.

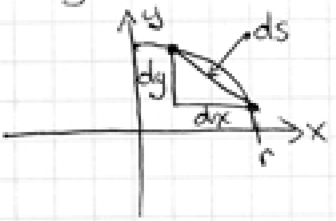
sketch:



Explicit (top half): $y = \sqrt{r^2 - x^2}$
$-r < x < r$.

Parametric: $x = r\cos t$ $0 < t < 2\pi$
(whole circle) $y = r\sin t$

Using Explicit function:



small bit of length $= ds$

$$= \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Circumference $= 2$(arclength of top half)

$$= 2\int_{-r}^{r} ds$$

$$= 4\int_0^r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{(using symmetry)}$$

$$\frac{dy}{dx} = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}}$$

$$\text{Circumference} = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

$$= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}}$$

To do this integral, we need to do a trig substitution.

Let $x = r\cos\theta$ $dx = -r\sin\theta\, d\theta$

$$\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2\cos^2\theta}$$

$$= r\sqrt{1 - \cos^2\theta}$$

$$= r\sin\theta.$$

$$\int \frac{dx}{\sqrt{r^2 - x^2}} = \int \frac{-r\sin\theta\, d\theta}{r\sin\theta}$$

$$= -\int d\theta = -\theta + C.$$

$$= -\arccos\left(\frac{x}{r}\right) + C$$

So Circumference $= 4r\left[-\arccos\left(\frac{x}{r}\right)\right]_0^r$

$$= 4r\left[-\arccos(1) + \arccos(0)\right]$$

$\arccos 1 = \omega \rightarrow \cos\omega = 1$
so $\omega = 0$.

$\arccos 0 = \omega \rightarrow \cos\omega = 0$
so $\omega = \pi/2$.

Finally, Circumference $= 4r\left[-0 + \pi/2\right]$
$= 2\pi r$

Using Implicit function:

sketch will be same as the one for explicit function.

We get Circumference $= \int_0^{2\pi} ds$

$$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -r\sin t$$

$$\frac{dy}{dt} = r\cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$
$$= r$$

So circumference $= \int_0^{2\pi} r\,dt = rt\Big|_0^{2\pi} = 2\pi r.$

That was easy.