How to Determine the Limit of a Sequence

A sequence can be given in a variety of ways. How you determine the limit depends on how the sequence is given. Here are the techniques:

- Sequence is given by $\{a_n\}$

  This is the simplest case, when you know a_n explicitly, and $\lim_{n\to\infty} a_n$ can be calculated directly.

  If l'Hospital's rule is used, you must switch to the continuous function for the derivative to make sense mathematically (derivatives need to be defined on function of continuous variables).

- Sequence is given recursively by $a_n = g(a_{n-1})$

  First, you must prove the limit exists. You can do this by showing that

  1. the sequence is monotonic (either increasing, or decreasing), and
  2. the sequence is bounded.

  You can use mathematical induction (see below) to prove these two things. If these two things are true, then the sequence is bounded and monotonic, so it converges by the monotonic sequence theorem.

  Now that you have shown the sequence converges, you can assume the sequence converges to L. Therefore,

  $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = L.$$

  Then, the limit of the sequence is found by the following analysis.

  $$a_n = g(a_{n-1}) \Rightarrow \lim_{n\to\infty} \Big(a_n = g(a_{n-1})\Big) \Rightarrow \lim_{n\to\infty} a_n = g(\lim_{n\to\infty} a_{n-1}) \Rightarrow L = g(L).$$

  Solve for L and pick the appropriate L.

- Sequence is given by $\{a_1, a_2, a_3, \ldots\}$ (general term is not provided)

  Here, you try to convert the sequence to one of the two previous forms by finding the pattern in the sequence.

Mathematical Induction

Mathematical induction is used to prove a result for all integers $n = 1, 2, 3, \ldots$. Here is the process.

1. Prove the result is true for $n = 1$.
2. Assume the result is true for $n = k$.
3. Prove the result is true for $n = k + 1$ (by using the assumed result in 2.).

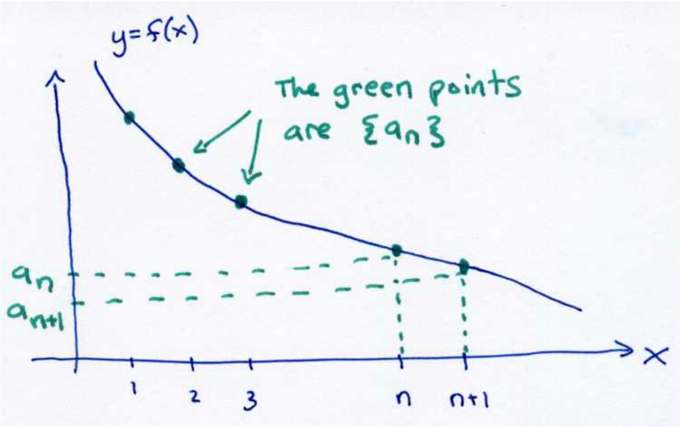
Once the above has been shown, then the result is true for all integers $n = 1, 2, 3, \ldots$ by mathematical induction.

Showing a Sequence is Decreasing by Switching to a Function on $\mathbb{R}$

You could also do this for increasing, with obvious modifications to the analysis. This process would be something you can do in some cases instead of using mathematical induction.

If you want to show a sequence $\{a_n\}$ is decreasing, you need to show $a_{n+1} < a_n$.

First, consider the function $f(x)$ that is given by $f(n) = a_n$. So the elements of the sequence lie on the curve $f(x)$, and if the sequence is decreasing it must look something like the following:



Notice that from the diagram we see $a_{n+1} < a_n$. From Calculus I, we know that the function $f(x)$ is decreasing if the derivative of f is less than zero, $f'(x) < 0$.

So, if you can show that $f'(x) < 0$ then you will have also shown that the sequence $\{a_n\}$ is decreasing.

Example Using Mathematical Induction For the sequence $a_1 = 2$, $a_{n+1} = \dfrac{1}{3 - a_n}$

(a) Show this is a decreasing sequence.
(b) Show this is a bounded sequence.
(c) Finally, determine the limit of the the sequence.

We can show the sequence is decreasing using induction to show $a_{n+1} < a_n, n = 1, 2, 3, \ldots$.

1. Show the statement is true for $n = 1$: $a_2 < a_1$.

$$a_2 = \frac{1}{3 - a_1} = 1 < 2 = a_1.$$

2. Assume the statement is true for $n = k$: $a_{k+1} < a_k$.

3. Show the statement is true for $n = k + 1$: $a_{k+2} < a_{k+1}$.

From 2. we have
$$\begin{aligned} &a_{k+1} < a_k \\ &-a_{k+1} > -a_k \\ &3 - a_{k+1} > 3 - a_k \\ &\frac{1}{3 - a_{k+1}} < \frac{1}{3 - a_k} \\ &a_{k+2} < a_{k+1} \end{aligned}$$

Therefore, $a_{n+1} < a_n, n = 1, 2, 3, \ldots$ by induction.

The sequence is decreasing (so it is monotonic).

We can show the sequence is bounded above using induction to show $a_n \leq 2, n = 1, 2, 3, \ldots$.

1. Show the statement is true for $n = 1$: $a_1 \leq 2$.

   $a_1 = 2 \leq 2$.

2. Assume the statement is true for $n = k$: $a_k \leq 2$.

3. Show the statement is true for $n = k + 1$: $a_{k+1} \leq 2$.

   From 2. we have
   $$\begin{aligned} & a_k \leq 2 \\ & -a_k \geq -2 \\ & 3 - a_k \geq 3 - 2 = 1 \\ & \frac{1}{3 - a_k} \leq 1 \\ & a_{k+1} \leq 1 \leq 2 \end{aligned}$$

   Therefore, $a_n \leq 2, n = 1, 2, 3, \ldots$ by induction.

For bounded below, we have that $a_{n+1} = 1/(3 - a_n) > 0$ since $a_n \leq 2$ for all n.

The sequence is therefore a bounded sequence.

The limit of the sequence therefore exists by the Monotonic Sequence Theorem.

To find the limit, we assume $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$.

$$\lim_{n\to\infty} \left[a_{n+1} = \frac{1}{3 - a_n} \right]$$
$$L = \frac{1}{3 - L} \Rightarrow 3L - L^2 = 1 \Rightarrow L = \frac{1}{2}(3 - \sqrt{5})$$

the other value is extraneous since it is greater that 2, and 2 is an upper bound for the sequence.