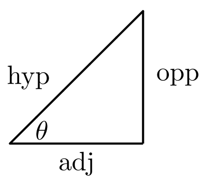
Appendix D has a trigonometric review. This material is meant to help you remember the values of the trig functions at special values, and help you see how the trig identities are related. You will not be tested on this material directly; you mainly need to have certain trig identities memorized, or know how to derive them if you need them. Remember– memorized means memorized correctly, not just that you are familiar with something! If you use an identity in class or on the homework that means it is important and might show up again.

Right Angle Triangles

The six basic trigonometric functions relate the angle θ to ratios of the length of the sides of the right triangle.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

From the definition of the trig functions, we immediately get the following identities:

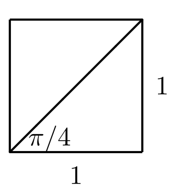
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

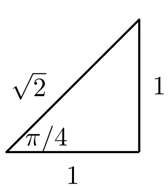
$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

For certain triangles, the trig functions of the angles can be found geometrically. These special triangles occur frequently enough that it is expected that you learn the value of the trig functions for the special angles.

A 45-45-90 Triangle

Consider the square given below.





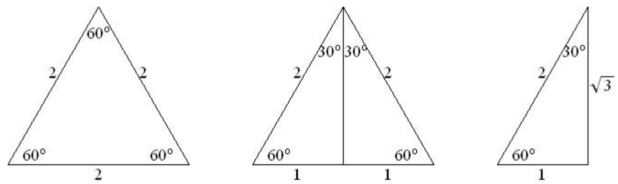
The angle here must be $\pi/4$ radians, since this triangle is half of a square of side length 1.

Now, we can write down all the trig functions for an angle of $\pi/4$ radians = 45 degrees:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2} \qquad \tan\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$
$$\cos\left(\frac{\pi}{4}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \sqrt{2} \qquad \cot\left(\frac{\pi}{4}\right) = \frac{1}{\tan\left(\frac{\pi}{4}\right)} = 1$$

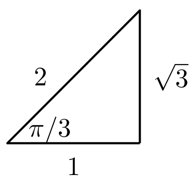
A 30-60-90 Triangle

Consider the equilateral triangle given below. Geometry allows us to construct a 30-60-90 triangle:



We can now determine the six trigonometric functions at two more angles!

$60^{\mathrm{o}} = \frac{\pi}{3}$ radians:

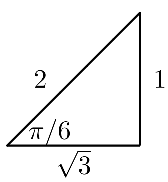


$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \cot\left(\frac{\pi}{3}\right) = \frac{1}{\tan\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}}$$

$30^{\mathrm{o}} = \frac{\pi}{6}$ radians:

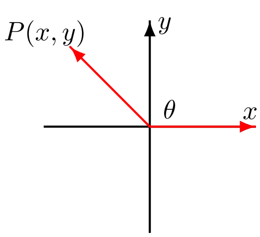


$$\sin\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \csc\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = 2$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} \qquad \cot\left(\frac{\pi}{6}\right) = \frac{1}{\tan\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

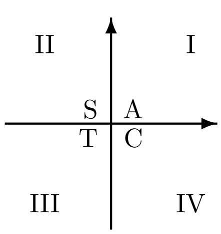
Obtuse Angles



If we label the point at the end of the terminal side as $P(x, y)$, and if we let $r = \sqrt{x^2 + y^2}$, we can construct the following relationships between the six trig functions and our diagram:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

$$\csc\theta = \frac{r}{y},\ y \neq 0, \qquad \sec\theta = \frac{r}{x},\ x \neq 0, \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$



The CAST diagram tells us the sign of sine, cosine and tangent in the quadrants.

Quadrant IV:	Cosine is positive, the other two are negative.
Quadrant I:	All are positive.
Quadrant II:	Sine is positive, the other two are negative.
Quadrant III:	Tangent is positive, the other two are negative.

Identities

You will need to be able to know the basic trig identities, or derive them. I recommend memorizing a few, and deriving others that you will need when necessary. I would memorize the following three:

$$\boxed{\cos^2 x + \sin^2 x = 1}$$

$$\boxed{\cos(u - v) = \cos u \cos v + \sin u \sin v}$$

$$\boxed{\sin(u + v) = \sin u \cos v + \cos u \sin v}$$

You will no doubt memorize others identities as you work with them, but if you do forget them you can derive them using ideas like what is below.

Deriving Other Identities

- Derive some identities from $\cos^2\theta + \sin^2\theta = 1$:

Divide by $\cos^2\theta$: $\dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} \longrightarrow \boxed{1 + \tan^2\theta = \sec^2\theta}$.

Divide by $\sin^2\theta$: $\dfrac{\cos^2\theta}{\sin^2\theta} + \dfrac{\sin^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta} \longrightarrow \boxed{\cot^2\theta + 1 = \csc^2\theta}$.

- Derive some identities from $\cos(u - v) = \cos u \cos v + \sin u \sin v$:

Evaluate at $u = \pi/2$: $\boxed{\cos(\pi/2 - v) = \sin v}$.

Evaluate at $u = 0$: $\boxed{\cos(-v) = \cos v}$.

Evaluate at $v = -v$: $\boxed{\cos(u + v) = \cos u \cos v - \sin u \sin v}$,

$\Rightarrow$ Now evaluate at $v = u$: $\boxed{\cos(2u) = \cos^2 u - \sin^2 u}$,

$\Rightarrow$ Now use $\sin^2 u + \cos^2 u = 1$: $\cos(2u) = \cos^2 u - 1 + \cos^2 u \longrightarrow \boxed{\cos^2 u = \frac{1}{2}(1 + \cos(2u))}$.

$\Rightarrow$ Now use $\sin^2 u + \cos^2 u = 1$: $\cos(2u) = 1 - \sin^2 u - \sin^2 u \longrightarrow \boxed{\sin^2 u = \frac{1}{2}(1 - \cos(2u))}$.

- Derive some identities from $\sin(u + v) = \sin u \cos v + \cos u \sin v$:

Evaluate at $v = u$: $\boxed{\sin(2u) = 2\sin u \cos u}$.

Evaluate at $u = \pi/2$: $\boxed{\sin(\pi/2 + v) = \cos v}$.

Evaluate at $v = -v$: $\boxed{\sin(u - v) = \sin u \cos v - \cos u \sin v}$. Now evaluate at $u = 0$: $\boxed{\sin(-v) = -\sin v}$

• Product to Sum Identities (very useful in certain types of trig integrals):

Add $\cos(u-v) = \cos u \cos v + \sin u \sin v$ and $\cos(u+v) = \cos u \cos v - \sin u \sin v$:

$$\boxed{\cos u \cos v = \frac{1}{2}\Big(\cos(u-v) + \cos(u+v)\Big)}$$

Subtract $\cos(u-v) = \cos u \cos v + \sin u \sin v$ from $\cos(u+v) = \cos u \cos v - \sin u \sin v$:

$$\boxed{\sin u \sin v = \frac{1}{2}\Big(\cos(u-v) - \cos(u+v)\Big)}$$

Add $\sin(u-v) = \sin u \cos v - \cos u \sin v$ and $\sin(u+v) = \sin u \cos v + \cos u \sin v$:

$$\boxed{\sin u \cos v = \frac{1}{2}\Big(\sin(u-v) + \sin(u+v)\Big)}$$