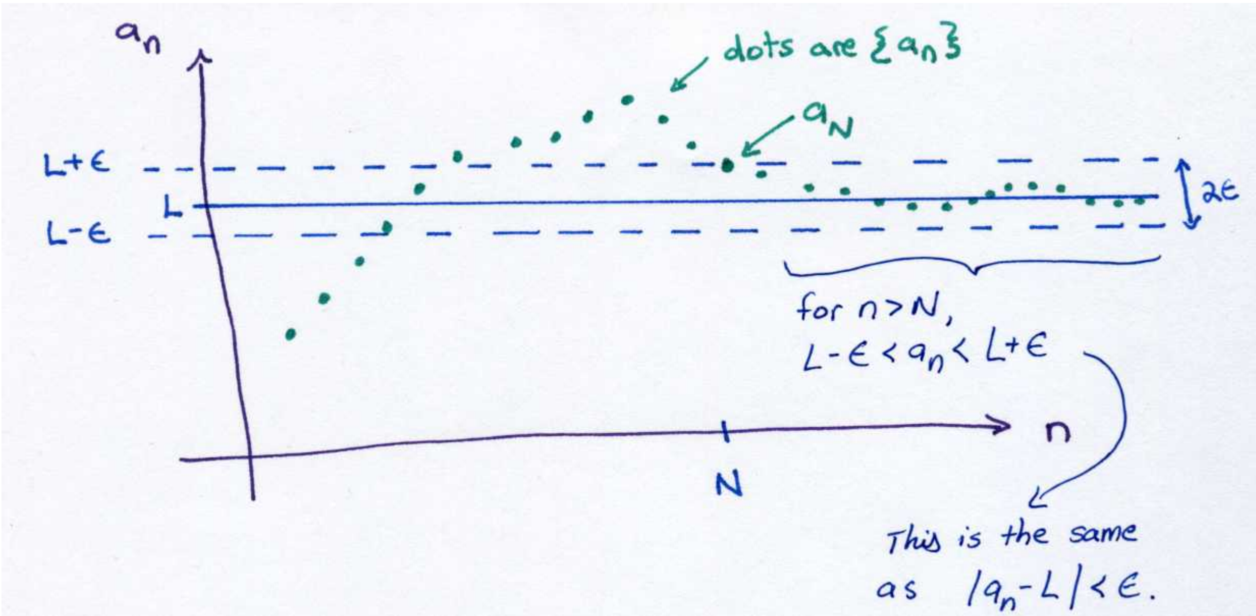
Definition of Convergence A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \epsilon$ whenever $n > N$. If $\lim_{n\to\infty} a_n$ exists, the sequence converges. Otherwise, it diverges.

Here is a sketch of what this is saying:



If we decrease the size of ϵ, we would have to increase the size of N to have $L - \epsilon < a_n < L + \epsilon$.

If we can do this for all $\epsilon > 0$, then we would say that the sequence $\{a_n\}$ has limit L, $\lim_{n\to\infty} a_n = L$.

If there was some $\epsilon > 0$ for which we could not do this, then we say the sequence $\{a_n\}$ is divergent.

Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

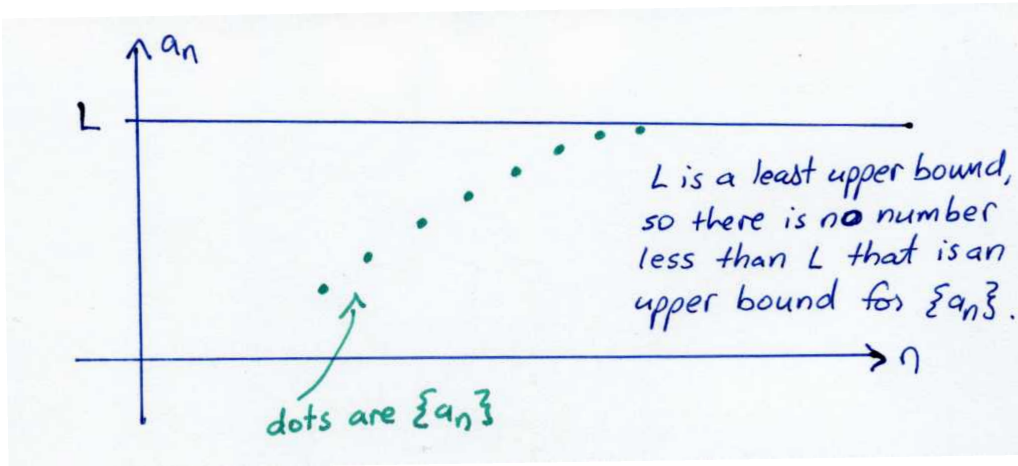
Proof. To show a sequence is is convergent, we must show that

for every $\epsilon > 0$ there is a corresponding integer N such that $\|a_n - L\| < \epsilon$ whenever $n > N$.

So our goal in this proof is to start with an unspecified increasing, bounded sequence and construct a series of steps that ends with the boxed statement above.

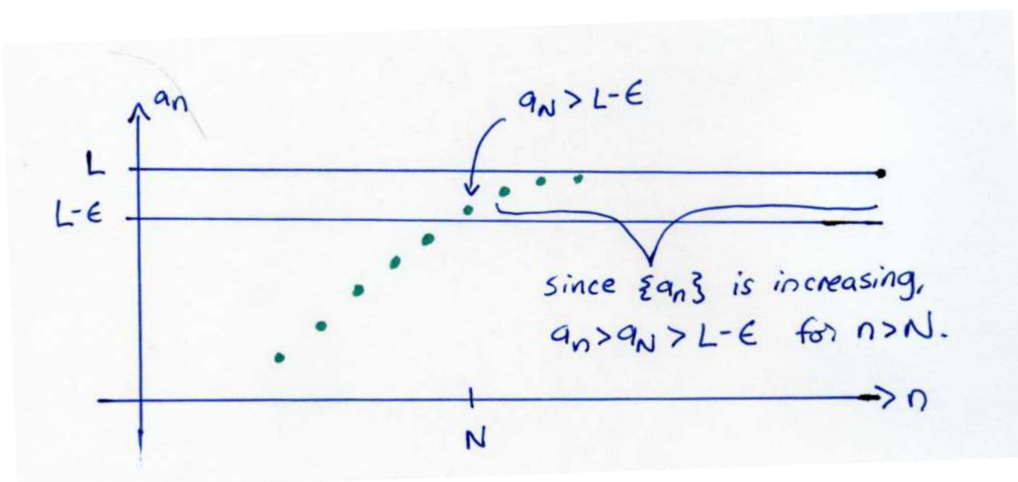
Assume $\{a_n\}$ is an increasing, bounded sequence.

Since $\{a_n\}$ is bounded, it has a Least Upper Bound L. We can draw a sketch that implies this fact, but our sketch can't be entirely accurate since we can't draw the entire infinite sequence in the sketch. So from the sketch it might look like you could find a number slightly smaller that L which would also be an upper bound, but that really isn't the case.



Therefore, if $\epsilon > 0$, $L - \epsilon$ is <u>not</u> an upper bound.

Therefore, $a_N > L - \epsilon$ for some integer N.



Since the sequence is increasing, $a_n > a_N$ for every $n > N$.

$$
\begin{array}{rl}
\text{Therefore} & a_n > a_N > L - \epsilon, \quad n > N \\
& a_n > L - \epsilon, \quad n > N \\
& a_n > L - \epsilon, \quad n > N \\
\text{Rearranging} & \epsilon > L - a_n, \quad n > N \\
& L - a_n < \epsilon, \quad n > N
\end{array}
$$

Since $a_n \le L$ (L is a least upper bound), we have $0 \le L - a_n < \epsilon$, $n > N$.

Therefore, $|L - a_n| < \epsilon$, $n > N$, which means that $\lim_{n\to\infty} a_n = L$ by the formal definition of convergence of a sequence.

□