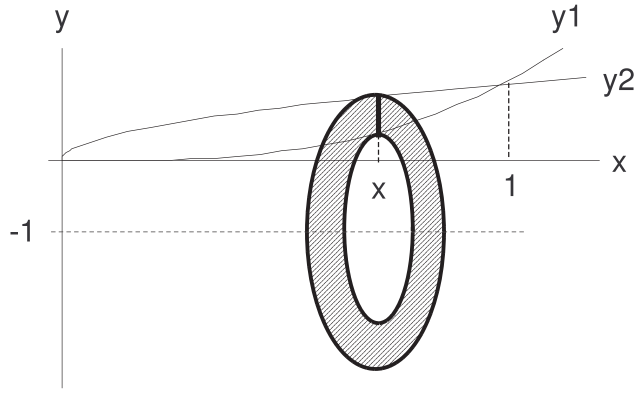
Example The area between $y = x^3$ and $y = \sqrt{x}$ is rotated about the line $y = -1$, creating a surface S. Find the volume of the surface S.

First, we want to sketch the situation. A good sketch will help you see how to solve the problem.



I have drawn my sketch slightly more general than it needs to be. The curves are $y_1 = x^3$ and $y_2 = \sqrt{x}$. The region being rotated is the area between the two curves. As x moves between 0 and 1, the small line in the region will sweep out the entire region. This tells us that we should integrate with respect to x, and we should have integration limits of 0 and 1.

The volume of the surface can be found by adding up the areas of all the washers (think of these as thin slices through the volume). Mathematically, how we "add up" the areas of all the washers is by integrating. So the volume we want to find is given by

$$\text{Volume of the surface } S = \int_0^1 (\text{area of the washer})\ dx.$$

The area of the washer is found via:

$$\begin{aligned}
\text{radius of outer circle} &= 1 + \sqrt{x} \\
\text{radius of inner circle} &= 1 + x^3 \\
\text{area of washer} &= \pi[(\text{outer radius})^2 - (\text{inner radius})^2] \\
&= \pi[(1 + \sqrt{x})^2 - (1 + x^3)^2] \\
&= \pi[x + 2\sqrt{x} - 2x^3 - x^6]
\end{aligned}$$

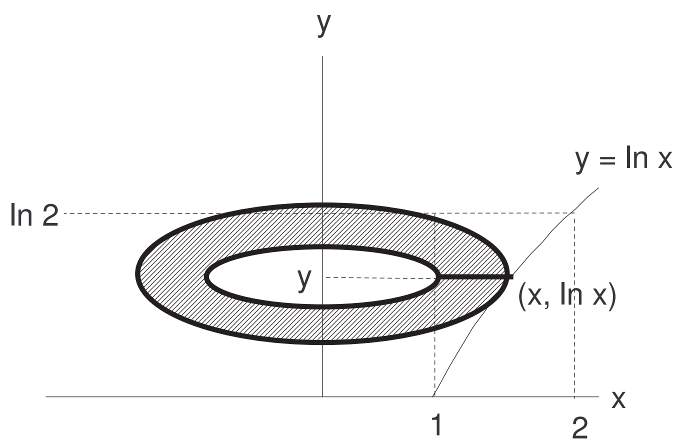
And so the volume we are asked to find is

$$V = \int_0^1 (\text{area of the washer})\ dx$$

$$\begin{aligned}
&= \int_0^1 \pi[x + 2\sqrt{x} - 2x^3 - x^6]\,dx \\
&= \pi\left[\frac{x^2}{2} + 2\frac{x^{(3/2)}}{3/2} - 2\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 \\
&= \pi\left[\frac{1}{2} + \frac{4}{3} - \frac{1}{2} - \frac{1}{7}\right] \\
&= \pi\left[\frac{4}{3} - \frac{1}{7}\right] \\
&= \frac{25\pi}{21}
\end{aligned}$$

Example The area above $1 \leq x \leq 2$, below $y = \ln 2$, and above $y = \ln x$ is rotated about the y-axis, creating a surface S. Find the volume of the surface S.

First, we want to sketch the situation. A good sketch will help you see how to solve the problem.



As y moves between 0 and $\ln 2$, the small line in the region will sweep out the entire region. This tells us that we should integrate with respect to y, and we should have integration limits of 0 and $\ln 2$.

The volume of the surface can be found by adding up the areas of all the washers (think of these as thin slices through the volume). Mathematically, how we "add up" the areas of all the washers is by integrating. So the volume we want to find is given by

$$\text{Volume of the surface } S = \int_0^{\ln 2} (\text{area of the washer})\ dy.$$

The area of the washer is found via:

$$\begin{aligned}
\text{radius of outer circle} &= x = e^y \\
\text{radius of inner circle} &= 1
\end{aligned}$$

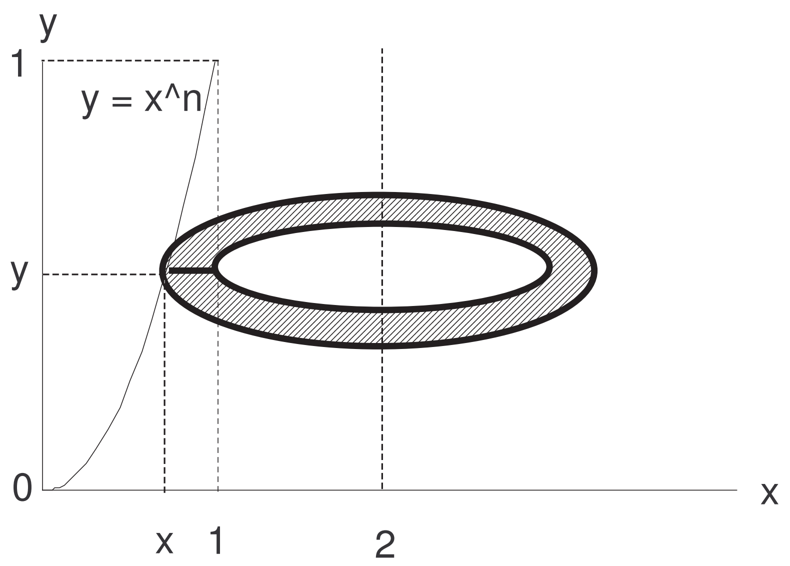
$$\begin{aligned}\text{area of washer} &= \pi[(\text{outer radius})^2 - (\text{inner radius})^2] \\ &= \pi[(e^y)^2 - (1)^2] \\ &= \pi[e^{2y} - 1]\end{aligned}$$

And so the volume we are asked to find is

$$\begin{aligned}V &= \int_0^{\ln 2} (\text{area of the washer})\ dy \\ &= \int_0^{\ln 2} \pi[e^{2y} - 1]\, dy \\ &= \pi\left[\frac{e^{2y}}{2} - y\right]_0^{\ln 2} \\ &= \pi\left[\left(\frac{1}{2}e^{2\ln 2} - \ln 2\right) - \left(\frac{1}{2}e^0 - 0\right)\right] \\ &= \pi\left[\frac{4}{2} - \ln 2 - \frac{1}{2}\right] \\ &= \pi\left[\frac{3}{2} - \ln 2\right]\end{aligned}$$

Example The area above $0 \leq x \leq 1$ and below $y = x^n$, $n \geq 1$, is rotated about the line $x = 2$, creating a surface S. Find the volume of the surface S.

First, we want to sketch the situation. A good sketch will help you see how to solve the problem.



As y moves between 0 and 1, the small line in the region will sweep out the entire region. This tells us that we should integrate with respect to y, and we should have integration limits of 0 and 1.

The volume of the surface can be found by adding up the areas of all the washers (think of these as thin slices through the volume). Mathematically, how we "add up" the areas of all the washers is by integrating. So the volume we want to find is given by

$$\text{Volume of the surface } S = \int_0^1 \text{ (area of the washer) } dy.$$

The area of the washer is found via:

$$\begin{aligned}
x + \text{ radius of outer circle} &= 2 \longrightarrow \text{ radius of outer circle } = 2 - x = 2 - y^{1/n} \\
\text{radius of inner circle} &= 1 \\
\text{area of washer} &= \pi[(\text{outer radius})^2 - (\text{inner radius})^2] \\
&= \pi[(2 - y^{1/n})^2 - (1)^2] \\
&= \pi[3 - 4y^{1/n} + y^{2/n}]
\end{aligned}$$

And so the volume we are asked to find is

$$\begin{aligned}
V &= \int_0^1 \text{ (area of the washer) } dy \\
&= \int_0^1 \pi[3 - 4y^{1/n} + y^{2/n}]\, dy \\
&= \pi\left[3y - 4\frac{y^{1/n+1}}{1/n+1} + \frac{y^{2/n+1}}{2/n+1}\right]_0^1 \\
&= \pi\left[3 - 4\frac{1}{1/n+1} + \frac{1}{2/n+1}\right] \\
&= \pi\left[3 - \frac{4n}{1+n} + \frac{n}{2+n}\right]
\end{aligned}$$