Example: Probability Distributions Find the value of A which makes

$$f(x) = \begin{cases} 0 & x > 1 \\ A & 0 \le x \le 1 \\ Ae^{\pi x} & x < 0 \end{cases}$$

a probability density function. Calculate the mean value of this probability density function.

NOTE: although this problem is longer than what I would typically give on a final exam, it is not more difficult than the types of problems I might give.

Since $f(x) \ge 0$ for all x, all we need to do to show this is a probability density function is calculate $\int_{-\infty}^{\infty} f(x)dx$ and choose the value of A which makes this equal to 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx \quad \text{since our function is piece-wise defined} \\
&= \int_{-\infty}^{0} Ae^{\pi x}\,dx + \int_{0}^{1} A\,dx + \int_{1}^{\infty} 0\,dx \\
&= A \lim_{a\to-\infty} \int_{a}^{0} e^{\pi x}\,dx + A + 0 \quad \text{Substitution: } \begin{array}{ll} u = \pi x & x = a \to u = \pi a \\ du = \pi\,dx & x = 0 \to u = 0 \end{array} \\
&= A + A \lim_{a\to-\infty} \frac{1}{\pi} \int_{\pi a}^{0} e^{u}\,du \\
&= A + \frac{A}{\pi} \lim_{a\to-\infty} e^{u}\Big|_{\pi a}^{0} \\
&= A + \frac{A}{\pi} \lim_{a\to-\infty} (1 - e^{\pi a}) \\
&= A + \frac{A}{\pi}(1 - 0) \\
&= A + \frac{A}{\pi}
\end{aligned}$$

We set this equal to one (which means we are forcing $\int_{-\infty}^{\infty} f(x)\,dx = 1$) and solve for A:

$$A + \frac{A}{\pi} = 1 \longrightarrow A = \frac{\pi}{1+\pi}.$$

To calculate the mean we need to calculate (I'll substitute for A at the end):

$$\begin{aligned}
\mu = \int_{-\infty}^{\infty} xf(x)\,dx &= \int_{-\infty}^{0} xf(x)\,dx + \int_{0}^{1} xf(x)\,dx + \int_{1}^{\infty} xf(x)\,dx \\
&= \int_{-\infty}^{0} Axe^{\pi x}\,dx + \int_{0}^{1} Ax\,dx + \int_{1}^{\infty} x \cdot 0\,dx \\
&= A \lim_{a\to-\infty} \int_{a}^{0} xe^{\pi x}\,dx + A\,\frac{x^2}{2}\Big|_{0}^{1} + 0 \\
&= \frac{A}{2} + A \lim_{a\to-\infty} \int_{a}^{0} xe^{\pi x}\,dx
\end{aligned}$$

Use parts to do the integral $\int xe^{\pi x}\,dx$:

$$\begin{array}{ll} u = x & dv = e^{\pi x}\,dx \\ du = dx & v = \int e^{\pi x}\,dx = \frac{1}{\pi}e^{\pi x} \end{array}$$

$$\begin{aligned} \int xe^{\pi x}\,dx &= \int u\,dv \\ &= uv - \int v\,du \\ &= \frac{x}{\pi}e^{\pi x} - \int \frac{1}{\pi}e^{\pi x}\,dx \\ &= \frac{x}{\pi}e^{\pi x} - \frac{1}{\pi^2}e^{\pi x} \end{aligned}$$

$$\begin{aligned} \mu &= \frac{A}{2} + A\lim_{a\to-\infty}\int_a^0 xe^{\pi x}\,dx \\ &= \frac{A}{2} + A\lim_{a\to-\infty}\left(\frac{x}{\pi}e^{\pi x} - \frac{1}{\pi^2}e^{\pi x}\right)\Bigg|_a^0 \\ &= \frac{A}{2} + A\lim_{a\to-\infty}\left[\left(0-\frac{1}{\pi^2}\right) - \left(\frac{a}{\pi}e^{\pi a} - \frac{1}{\pi^2}e^{\pi a}\right)\right] \\ &= \frac{A}{2} - A\lim_{a\to-\infty}\frac{1}{\pi^2} - A\lim_{a\to-\infty}\frac{a}{\pi}e^{\pi a} + A\lim_{a\to-\infty}\frac{1}{\pi^2}e^{\pi a} \\ &= \frac{A}{2} - \frac{A}{\pi^2} - A\lim_{a\to-\infty}\frac{a}{\pi}e^{\pi a} + A\cdot 0 \\ &= \frac{A}{2} - \frac{A}{\pi^2} - A\lim_{a\to-\infty}\frac{a}{\pi}e^{\pi a} \end{aligned}$$

To evaluate this last limit, we need to look at it more closely:

$$\begin{aligned} \lim_{a\to-\infty}\frac{a}{\pi}e^{\pi a} &\to (-\infty)\cdot(0) \quad \text{indeterminate product} \\ &= \frac{1}{\pi}\lim_{a\to-\infty}\frac{a}{e^{-\pi a}} \to \frac{-\infty}{\infty} \quad \text{indeterminate quotient, so use L'Hospital's Rule} \\ &= \frac{1}{\pi}\lim_{a\to-\infty}\frac{\frac{d}{da}a}{\frac{d}{da}e^{-\pi a}} \\ &= \frac{1}{\pi}\lim_{a\to-\infty}\frac{1}{(-\pi)e^{-\pi a}} \\ &= -\frac{1}{\pi^2}\lim_{a\to-\infty}\frac{1}{e^{-\pi a}} \\ &= -\frac{1}{\pi^2}\lim_{a\to-\infty}e^{\pi a} \\ &= -\frac{1}{\pi^2}(0) \\ &= 0 \end{aligned}$$

So the mean for the probability distribution is

$$\mu = \frac{A}{2} - \frac{A}{\pi^2} = A\left(\frac{1}{2} - \frac{1}{\pi^2}\right) = \frac{\pi}{1+\pi}\left(\frac{1}{2} - \frac{1}{\pi^2}\right).$$

Example: Orthogonal Trajectories Find the orthogonal trajectories to the family of curves

$$y = \frac{k}{1+x^2}.$$

The differential equation satisfied by the original family of curves is found by differentiating:

$$\begin{aligned}
y &= \frac{k}{1+x^2} \\
\frac{d}{dx}(y &= \frac{k}{1+x^2}) \\
\frac{dy}{dx} &= k\frac{d}{dx}\left(\frac{1}{1+x^2}\right) \\
&= -k(1+x^2)^{-2}(2x) \\
&= -2xk\frac{1}{(1+x^2)^2} \quad \text{use the original equation to eliminate } k \\
&= -2x(y(1+x^2))\frac{1}{(1+x^2)^2} \\
\frac{dy}{dx} &= -\frac{2xy}{(1+x^2)}
\end{aligned}$$

The differential equation satisfied by the orthogonal trajectories is therefore given by (derivatives must be negative reciprocals):

$$\begin{aligned}
-\frac{dx}{dy} &= -\frac{2xy}{(1+x^2)} \\
\frac{(1+x^2)}{x}\,dx &= 2y\,dy \quad \text{(separate)} \\
\int \frac{(1+x^2)}{x}\,dx &= \int 2y\,dy \quad \text{(integrate)} \\
\int \frac{1}{x}\,dx + \int x\,dx &= \int 2y\,dy \\
\ln|x| + \frac{x^2}{2} + c_1 &= y^2 + c_2 \\
\ln|x| + \frac{x^2}{2} &= y^2 + c \quad (c = c_2 - c_1)
\end{aligned}$$

The orthogonal trajectories are given implicitly by the equation $\ln|x| + \frac{x^2}{2} = y^2 + c$, where c is a constant.

Example: Orthogonal Trajectories Find the orthogonal trajectories to the family of curves

$$y^2 = \frac{k}{1+x}.$$

The differential equation satisfied by the original family of curves is found by differentiating:

$$\begin{aligned}
y^2 &= \frac{k}{1+x} \\
\frac{d}{dx}(y^2 &= \frac{k}{1+x}) \\
2y\frac{dy}{dx} &= k\frac{d}{dx}\left(\frac{1}{1+x}\right) \\
&= -k(1+x)^{-2} \\
&= -k\frac{1}{(1+x)^2} \quad \text{use the original equation to eliminate } k \\
&= -(y^2(1+x))\frac{1}{(1+x)^2} \\
\frac{dy}{dx} &= -\frac{y}{2(1+x)}
\end{aligned}$$

The differential equation satisfied by the orthogonal trajectories is therefore given by (derivatives must be negative reciprocals):

$$\begin{aligned}
-\frac{dx}{dy} &= -\frac{y}{2(1+x)} \\
(1+x)\,dx &= \frac{1}{2}y\,dy \quad \text{(separate)} \\
\int (1+x)\,dx &= \int \frac{1}{2}y\,dy \quad \text{(integrate)} \\
x + \frac{x^2}{2} + c_1 &= \frac{y^2}{4} + c_2 \\
x + \frac{x^2}{2} &= \frac{y^2}{4} + c \quad (x = c_2 - c_1)
\end{aligned}$$

The orthogonal trajectories are given implicitly by the equation $x + \frac{x^2}{2} = \frac{y^2}{4} + c$, where c is a constant.