This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. To prepare for the test:

- review homework,
- review WeBWorK,
- review examples from the text,
- review examples from class, and
- use the Study Guide.

The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words.

There will be five problems on the test, some may have two parts. You will have 100 minutes to complete the test. You may not use *Mathematica* or calculators on this test.

The following Useful Information will be provided on the test:

$$\begin{aligned}\cos^2 x + \sin^2 x &= 1\\ \cos(x+y) &= \cos x \cos y - \sin x \sin y\\ \sin(x+y) &= \sin x \cos y + \cos x \sin y\end{aligned}$$

Questions

1. Sketch the region enclosed by the curves $y = 1 - x^2, y = x^2$. Find the area of the region.

2. Sketch the region enclosed by the curves $y = \cos x, y = 1 - 2x/\pi, 0 \le x \le \pi$. Find the area of the region.

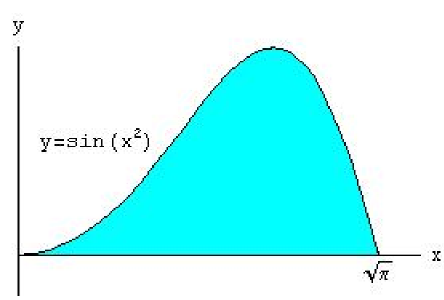
3. The region below $y = x^2$ and above the interval $0 \le x \le 2$ is rotated about the line $x = -1$, creating a solid S. Sketch the situation, and find the volume of the solid.

4. The region enclosed by the curves $y = x^2$ and $y = x$ is rotated about the line $y = 2$. Find the volume of the resulting solid.

5. What does the Mean Value Theorem for Integrals say? What is its geometric interpretation?

6. Find the numbers $b \ge 0$ such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

7. The region under the curve $y = \sin(x^2)$ and above $0 \le x \le \sqrt{\pi}$ (see diagram below) is rotated about the y-axis, creating a solid. Find the volume of this solid using cylindrical shells.



8. Derive the formula for integration by parts, beginning with the formula for the derivative of a product of functions.

9. First make a substitution, and then use parts to evaluate the integral

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta$$

10. Prove the following integration formula using trigonometric identities to evaluate the integral.

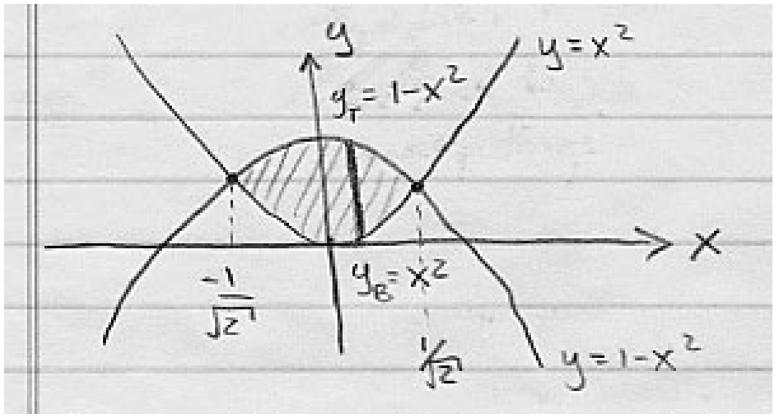
$$\int \cos^2 x \sin^3 x\, dx = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

11. Evaluate the integral using a trig substitution

$$\int \frac{1}{t^3\sqrt{t^2-1}}\, dt$$

Solutions

1. First, sketch the situation. Both these curves are parabolas, one opening up with a minimum of 0 at $x = 0$ ($y = x^2$); the other opening down with a maximum of 1 at $x = 0$ ($y = 1 - x^2$).



Over the region we have $y_T = 1 - x^2$ and $y_B = x^2$.

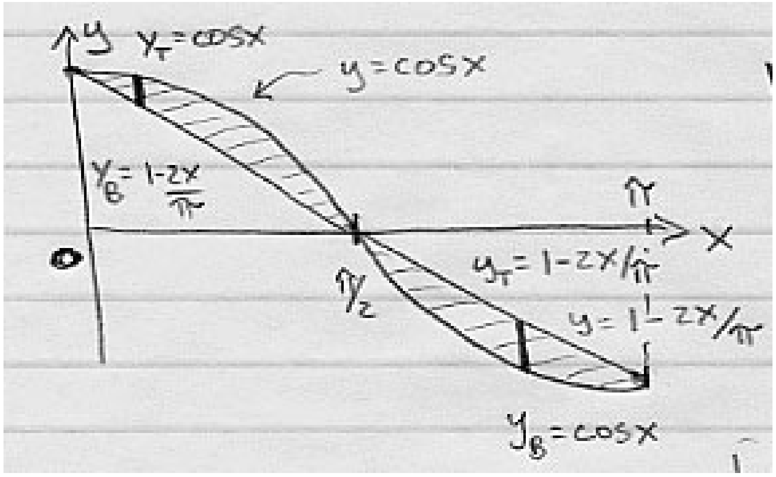
Integration limits: Solve for the point of intersection:

$$\begin{aligned} y_T &= y_B \\ 1 - x^2 &= x^2 \\ x^2 &= 1/2 \\ x &= \pm\sqrt{1/2} \end{aligned}$$

Integration limits: $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$.

$$\begin{aligned} \text{Area} &= \int_a^b (y_T - y_B)\,dx \\ &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - x^2 - x^2)\,dx \\ &= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - 2x^2)\,dx \\ &= 2\int_0^{1/\sqrt{2}} (1 - 2x^2)\,dx \\ &= 2\left(x - \frac{2}{3}x^3\right)_0^{1/\sqrt{2}} \\ &= 2\left(\frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3\right) - 0 \\ &= \frac{4}{3\sqrt{2}} \end{aligned}$$

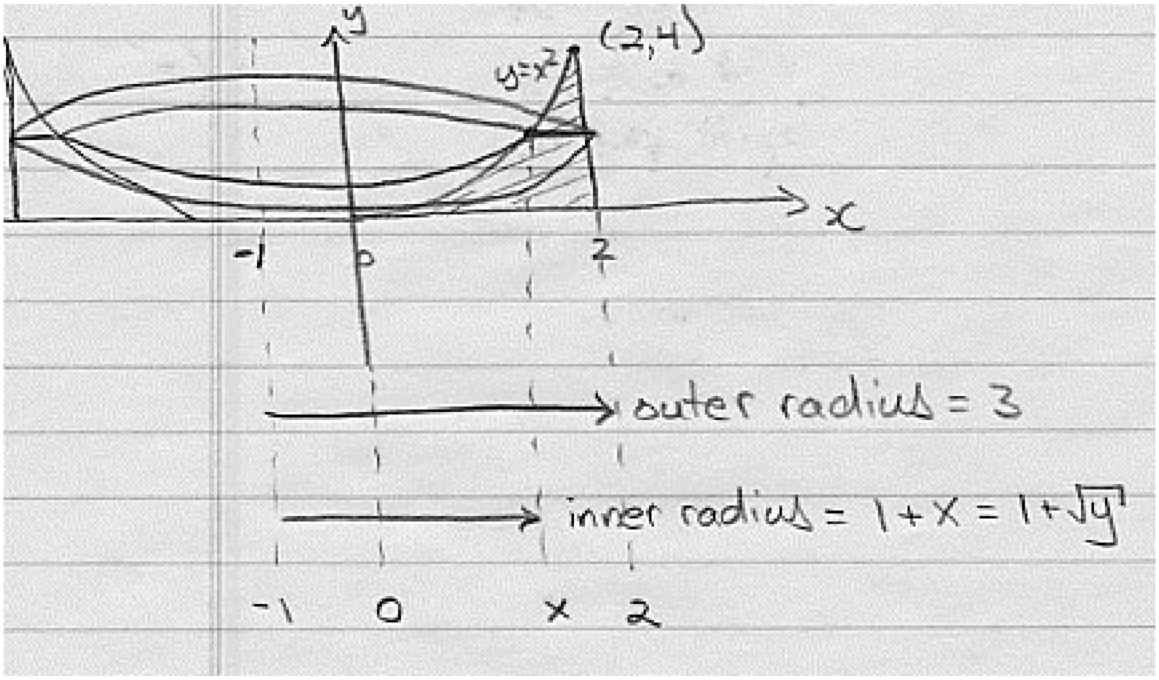
2. First, sketch the situation. We are told something about the region, $0 \leq x \leq \pi$, so that will help us. We can sketch the cosine function in that region. The other curve, $y = 1 - 2x/\pi$, is linear, and has slope $-2/\pi$. At $x = 0$ it is $+1$, at $x = \pi/2$ it is 0, and at $x = \pi$ it is -1. That works out well!



Over the region $0 \leq x \leq \pi$, the y_T and y_B change. This means we will have to split our integral up into two parts. We could also use symmetry to simplify the integral, since the area between $0 \leq x \leq \pi/2$ is the same as the area between $\pi/2 \leq x \leq \pi$.

$$\begin{aligned}
\text{Area} &= \int_a^b (y_T - y_B)\,dx \\
&= \int_0^{\pi/2} (y_T - y_B)\,dx + \int_{\pi/2}^{\pi} (y_T - y_B)\,dx \\
&= 2\int_0^{\pi/2} (y_T - y_B)\,dx \\
&= 2\int_0^{\pi/2} (\cos x - 1 + 2x/\pi)\,dx \\
&= 2(\sin x - x + x^2/\pi)_0^{\pi/2} \\
&= 2(\sin \pi/2 - \pi/2 + (\pi/2)^2/\pi) - 2(\sin 0 - 0 + (0)^2/\pi) \\
&= 2\left(1 - \frac{\pi}{2} + \frac{\pi}{4}\right) \\
&= 2 - \frac{\pi}{2}
\end{aligned}$$

3. First, sketch the situation:



Integration limits: if $0 < y < 4$, the washer sweeps out the volume. We integrate with respect to y.
Radius of inner circle $= 1 + x = 1 + \sqrt{y}$.
Radius of outer circle $= 3$.
Area of washer $= \pi[(\text{radius outer circle})^2 - (\text{radius of inner circle})^2] = \pi[9 - (1+\sqrt{y})^2] = \pi[9 - (1 + 2\sqrt{y} + y)] = \pi[8 - 2\sqrt{y} - y]$.

$$
\begin{aligned}
\text{Volume} &= \int_c^d (\text{area of washer})\, dy \\
&= \pi \int_0^4 (8 - 2\sqrt{y} - y)\, dy \\
&= \pi \left(8y - 2\frac{y^{3/2}}{3/2} - \frac{y^2}{2} \right)_0^4 \\
&= \pi \left(8(4) - \frac{4(4)^{3/2}}{3} - \frac{(4)^2}{2} \right) - 0 \\
&= \pi \left(32 - \frac{4(8)}{3} - 8 \right) \\
&= \pi \left(24 - \frac{32}{3} \right) \\
&= \frac{40\pi}{3}
\end{aligned}
$$

This problem could also be solved using cylindrical shells.

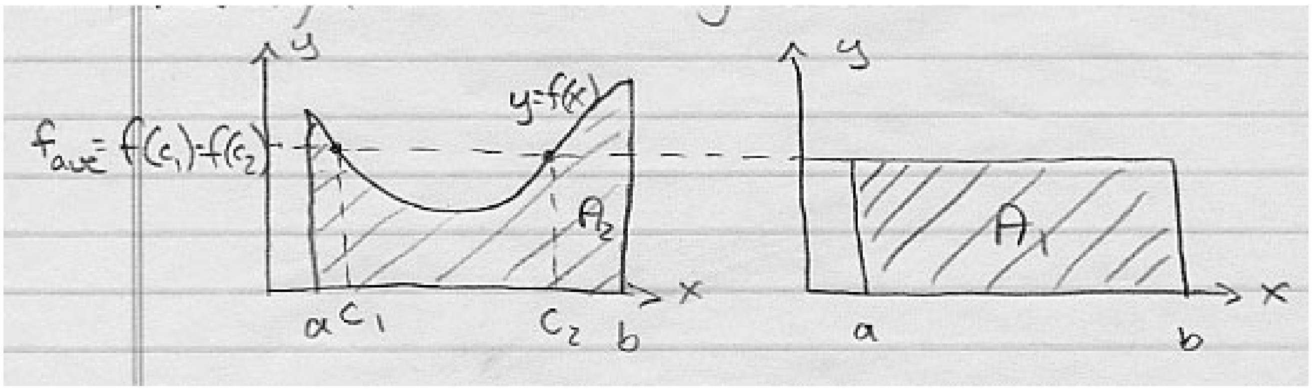
4. Example 5 on Page 444 of the text.

5. The Mean Value Theorem for Integrals says

$$\int_a^b f(x)\,dx = f(c)(b-a),$$

where $f(c)$ is the average value of $f(x)$ on the interval $x \in [a,b]$. The function $f(x)$ must be continuous on $[a,b]$, and there will be at least one $c \in [a,b]$ for which this is true.

Geometrically, if we assume $f(x) \geq 0$ for $x \in [a,b]$, this might look something like the following:



The areas A_1 and A_2 are the same. The "mountains" are cut down and used to fill in the valleys, so we are left with a flat plain of height $f(c)$.

For the diagram above, there are actually two $c \in [a,b]$ for which $f(c) = f_{ave}$.

6.

$$\begin{aligned} f_{ave} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{b-0}\int_0^b (2+6x-3x^2)\,dx \\ &= \frac{1}{b}(2x+3x^2-x^3)_0^b \\ &= \frac{1}{b}[(2b+3b^2-b^3)-0] \\ &= \frac{b}{b}(2+3b-b^2) \end{aligned}$$

Now, we can say $\dfrac{b}{b} = 1$ as long as $b \neq 0$. We need to check $b = 0$ separately, since it is going to be excluded in what follows.

If $b = 0$, then the interval is $[0,0]$, and $f_ave = f(0) = 2 \neq 3$, so $b = 0$ is not going to give us an average value of the function of 3.

Back to our previous work:

$$f_{ave} = 2 + 3b - b^2, \quad b \neq 0$$

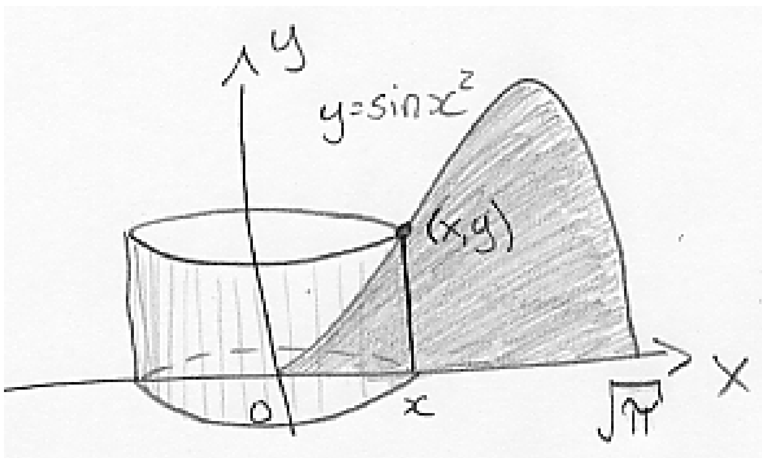
Now, we want to find out what values of b make $f_{ave} = 3$. Therefore, we must solve $2 + 3b - b^2 = 3$ for b.

$$\begin{aligned} 2 + 3b - b^2 &= 3 \\ b^2 - 3b + 1 &= 0 \quad \text{(use quadratic formula)} \\ b &= \frac{+3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

Both of these are greater than zero, so the two values of b for which $f_{ave} = 3$ are

$$b = \frac{3+\sqrt{5}}{2}, \quad b = \frac{3-\sqrt{5}}{2}.$$

7. Begin with a sketch.



Integration limits: if $0 < x < \sqrt{\pi}$, the cylinder sweeps out the total volume. We integrate with respect to x.
Cylinder height $= y = \sin x^2$.
Cylinder radius $= x$.
Cylinder circumference $= 2\pi \times \text{(radius)} = 2\pi x$.
Cylinder surface area $= 2\pi x \sin x^2$.

$$\begin{aligned} \text{Volume} &= \int_a^b \text{(area of cylinder)}\, dx \\ &= \int_0^{\sqrt{\pi}} 2\pi x \sin x^2 \, dx \qquad \text{Substitution: } \begin{array}{l} u = x^2 \\ du = 2x\, dx \\ \text{change limits:} \\ \text{when } x = 0 \to u = 0 \\ \text{when } x = \sqrt{\pi} \to u = \pi \end{array} \\ &= \pi \int_0^{\pi} \sin u \, du \\ &= -\pi \cos u \big|_0^{\pi} \\ &= -\pi (\cos \pi - \cos 0) \\ &= -\pi(-1-1) \\ &= 2\pi \end{aligned}$$

8. Start with the product form for derivative:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Write this in terms of differentials:

$$d[f(x)g(x)] = [f(x)g'(x) + f'(x)g(x)]\, dx$$

Integrate:

$$\int d[f(x)g(x)] = \int [f(x)g'(x) + f'(x)g(x)]\, dx$$

$$f(x)g(x) = \int f(x)g'(x)\, dx + \int f'(x)g(x)\, dx$$

Rewrite as:

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx$$

Rewrite again, using the substitutions:

$$\begin{array}{ll} u = f(x) & v = g(x) \\ du = f'(x)dx & dv = g'(x)dx \end{array}$$

And we get the formula for integration by parts:

$$\int u\, dv = uv - \int v\, du$$

9. First, the substitution:

$$x = \theta^2, \quad dx = 2\theta d\theta$$

Since this is a definite integral, we can change the limits of integration now:
when $\theta = \sqrt{\pi/2} \longrightarrow x = \pi/2$,
when $\theta = \sqrt{\pi} \longrightarrow x = \pi$.

We can write the integral as

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} x \cos x\, dx$$

Now, we use parts, and we choose:

$$\begin{array}{ll} u = x & dv = \cos x\, dx \\ du = dx & v = \sin x \end{array}$$

$$
\begin{aligned}
\int x \cos x \, dx &= \int u \, dv \\
&= uv - \int v \, du \text{ (Parts)} \\
&= x \sin x - \int \sin x \, dx \\
&= x \sin x - (-\cos x) \\
&= x \sin x + \cos x
\end{aligned}
$$

We are working with a definite integral, so we don't need to include the constant of integration.

The integral therefore becomes:

$$
\begin{aligned}
\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2) \, d\theta &= \frac{1}{2} \int_{\pi/2}^{\pi} x \cos x \, dx \\
&= \frac{1}{2} \left[x \sin x + \cos x \right]_{\pi/2}^{\pi} \\
&= \frac{1}{2} \left[\pi \sin \pi + \cos pi \right] - \frac{1}{2} \left[\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{pi}{2} \right] \\
&= \frac{1}{2} \left[\pi(0) + (-1) \right] - \frac{1}{2} \left[\frac{\pi}{2}(1) + (0) \right] \\
&= -\frac{1}{2} - \frac{\pi}{4}
\end{aligned}
$$

10. Since the integral involves an odd power of the sine, we will start by using the trig identity $\sin^2 x = 1 - \cos^2 x$ to rewrite the integral as

$$
\begin{aligned}
\int \cos^2 x \sin^3 x \, dx &= \int \cos^2 x \sin x \sin^2 x \, dx \\
&= \int \cos^2 x \sin x (1 - \cos^2 x) \, dx \\
&= \int \cos^2 x (1 - \cos^2 x) \sin x \, dx \quad \text{Substitution: } \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \\
&= -\int u^2 (1 - u^2) \, du \\
&= -\int (u^2 - u^4) \, du \\
&= -\left(\frac{u^3}{3} - \frac{u^5}{5} \right) + C \\
&= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C
\end{aligned}
$$

11. We cannot use a simple substitution to do this integral, so we try a trig substitution instead. The integrand contains a $\sqrt{t^2-a^2}$, with $a=1$, so we should use the substitution

$$\begin{aligned} t &= 1\cdot\sec\theta \\ dt &= \sec\theta\tan\theta\,d\theta \\ \sqrt{t^2-1} &= \sqrt{\sec^2\theta-1}=\sqrt{\tan^2\theta}=\tan\theta \end{aligned}$$

The integral becomes

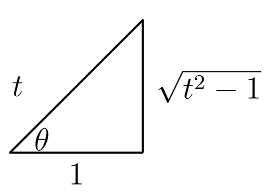
$$\begin{aligned} \int\frac{1}{t^3\sqrt{t^2-1}}\,dt &= \int\frac{\sec\theta\tan\theta\,d\theta}{\sec^3\theta\tan\theta} \\ &= \int\frac{d\theta}{\sec^2\theta} \\ &= \int\cos^2\theta\,d\theta \end{aligned}$$

This is even power of cosine, so we should use the 1/2-angle trig identity to do the integral.

$$\begin{aligned} \int\cos^2\theta\,d\theta &= \int\frac{1}{2}[1+\cos 2\theta]\,d\theta \\ &= \frac{1}{2}\int d\theta+\frac{1}{2}\int\cos 2\theta\,d\theta \quad \text{Substitution: } \begin{array}{c} x=2\theta \\ dx=2\,d\theta \end{array} \\ &= \frac{1}{2}\theta+\frac{1}{4}\int\cos x\,dx \\ &= \frac{1}{2}\theta+\frac{1}{4}\sin x+C \\ &= \frac{1}{2}\theta+\frac{1}{4}\sin 2\theta+C \\ &= \frac{1}{2}\theta+\frac{1}{4}2\sin\theta\cos\theta+C=\frac{1}{2}\theta+\frac{1}{2}\sin\theta\cos\theta+C \end{aligned}$$

Now we only need to back substitute for θ, to get the final answer in terms of t. Construct the diagram that will help us back substitute the θ:

$$\sec\theta=\frac{t}{1}\to\cos\theta=\frac{1}{t}$$



$$\sin\theta=\frac{\sqrt{t^2-1}}{t},\quad \cos\theta=\frac{1}{t},\quad \theta=\arccos\left(\frac{1}{t}\right)$$

And now we can finish the back substitution:

$$\int\frac{1}{t^3\sqrt{t^2-1}}\,dt=\frac{1}{2}\arccos\left(\frac{1}{t}\right)+\frac{1}{2}\frac{\sqrt{t^2-1}}{t}\frac{1}{t}+C=\frac{1}{2}\arccos\left(\frac{1}{t}\right)+\frac{1}{2}\frac{\sqrt{t^2-1}}{t^2}+C$$