This is a set of practice test problems for Chapter 7. This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words. Make sure to also study any homework problems, problems done in class, and problems that are performed in the textbook.

There will be five problems on the test. You will have 65 minutes to complete the test. You may not use *Mathematica* or calculators on this test.

Questions

1. Derive the formula for integration by parts, beginning with the formula for the derivative of a product of functions.

2. First make a substitution, and then use parts to evaluate the integral

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta$$

3. Prove the following integration formula using trigonometric identities to evaluate the integral.

$$\int \cos^2 x \sin^3 x\, dx = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$$

4. Evaluate the integral

$$\int \frac{1}{t^3\sqrt{t^2-1}}\, dt$$

5. Use partial fractions to evaluate the following integral. Your final answer should be simplified to the form of a natural logarithm of a rational number.

$$\int_3^5 \frac{x-9}{x^2+3x-10}\, dx$$

6. Use partial fractions to show that the following integral can be expressed as

$$\int \frac{3x^2-2}{(x^2+1)(x+1)}\, dx = \frac{5}{2}\int \frac{x}{x^2+1}\, dx - \frac{5}{2}\int \frac{1}{x^2+1}\, dx + \frac{1}{2}\int \frac{1}{x+1}\, dx$$

Then, explain in a couple of words how you would calculate each of the three integrals. That is, is it a standard form (power rule, inverse sine, logarithmic, etc), or would you use trig substitution to do it, or u substitution, parts, etc. **Do not evaluate the integrals**

7. Evaluate the integral

$$\int_0^\infty e^{-\pi x}\, dx$$

Solutions

1. Start with the product form for derivative:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Write this in terms of differentials:

$$d[f(x)g(x)] = [f(x)g'(x) + f'(x)g(x)]\,dx$$

Integrate:

$$\int d[f(x)g(x)] = \int [f(x)g'(x) + f'(x)g(x)]\,dx$$

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

Rewrite as:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Rewrite again, using the substitutions:

$$\begin{array}{ll} u = f(x) & v = g(x) \\ du = f'(x)dx & dv = g'(x)dx \end{array}$$

And we get the formula for integration by parts:

$$\int u\,dv = uv - \int v\,du$$

2. First, the substitution:

$$x = \theta^2, \quad dx = 2\theta d\theta$$

Since this is a definite integral, we can change the limits of integration now:
when $\theta = \sqrt{\pi/2} \longrightarrow x = \pi/2$,
when $\theta = \sqrt{\pi} \longrightarrow x = \pi$.

We can write the integral as

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$$

Now, we use parts, and we choose:

$$\begin{array}{ll} u = x & dv = \cos x\,dx \\ du = dx & v = \sin x \end{array}$$

$$\begin{aligned} \int x\cos x\,dx &= \int u\,dv \\ &= uv - \int v\,du \text{ (Parts)} \\ &= x\sin x - \int \sin x\,dx \\ &= x\sin x - (-\cos x) \\ &= x\sin x + \cos x \end{aligned}$$

We are working with a definite integral, so we don't need to include the constant of integration.

The integral therefore becomes:

$$\begin{aligned}
\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\, d\theta &= \frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\, dx \\
&= \frac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi} \\
&= \frac{1}{2}\left[\pi\sin\pi + \cos pi\right] - \frac{1}{2}\left[\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{pi}{2}\right] \\
&= \frac{1}{2}\left[\pi(0) + (-1)\right] - \frac{1}{2}\left[\frac{\pi}{2}(1) + (0)\right] \\
&= -\frac{1}{2} - \frac{\pi}{4}
\end{aligned}$$

3. Since the integral involves an odd power of the sine, we will start by using the trig identity $\sin^2 x = 1 - \cos^2 x$ to rewrite the integral as

$$\begin{aligned}
\int \cos^2 x \sin^3 x\, dx &= \int \cos^2 x \sin x \sin^2 x\, dx \\
&= \int \cos^2 x \sin x (1 - \cos^2 x)\, dx \\
&= \int \cos^2 x (1 - \cos^2 x) \sin x\, dx \quad \text{Substitution: } \begin{array}{l} u = \cos x \\ du = -\sin x\, dx \end{array} \\
&= -\int u^2(1 - u^2)\, du \\
&= -\int (u^2 - u^4)\, du \\
&= -\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C \\
&= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C
\end{aligned}$$

4. We cannot use a simple substitution to do this integral, so we try a trig substitution instead. The integrand contains a $\sqrt{t^2 - a^2}$, with $a = 1$, so we should use the substitution

$$\begin{aligned}
t &= 1 \cdot \sec\theta \\
dt &= \sec\theta\tan\theta\, d\theta \\
\sqrt{t^2 - 1} &= \sqrt{\sec^2\theta - 1} \\
&= \sqrt{\tan^2\theta} \\
&= \tan\theta
\end{aligned}$$

The integral becomes

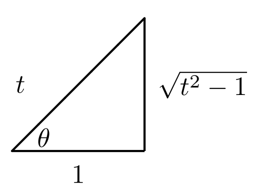
$$\begin{aligned}
\int \frac{1}{t^3\sqrt{t^2 - 1}}\, dt &= \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^3\theta\tan\theta} \\
&= \int \frac{d\theta}{\sec^2\theta} \\
&= \int \cos^2\theta\, d\theta
\end{aligned}$$

This is even power of cosine, so we should use the 1/2-angle trig identity to do the integral.

$$\begin{aligned}
\int \cos^2\theta \, d\theta &= \int \frac{1}{2}[1+\cos 2\theta]\, d\theta \\
&= \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta \, d\theta \quad \text{Substitution: } \begin{array}{l} x = 2\theta \\ dx = 2\,d\theta \end{array} \\
&= \frac{1}{2}\theta + \frac{1}{4}\int \cos x \, dx \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin x + C \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\
&= \frac{1}{2}\theta + \frac{1}{4} 2\sin\theta\cos\theta + C \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C
\end{aligned}$$

Now we only need to back substitute for θ, to get the final answer in terms of t. Construct the diagram that will help us back substitute the θ:

$$\sec\theta = \frac{t}{1} \rightarrow \cos\theta = \frac{1}{t}$$



$$\sin\theta = \frac{\sqrt{t^2-1}}{t}, \quad \cos\theta = \frac{1}{t}, \quad \theta = \arccos\left(\frac{1}{t}\right)$$

And now we can finish the back substitution:

$$\int \frac{1}{t^3\sqrt{t^2-1}}\, dt = \frac{1}{2}\arccos\left(\frac{1}{t}\right) + \frac{1}{2}\frac{\sqrt{t^2-1}}{t}\frac{1}{t} + C = \frac{1}{2}\arccos\left(\frac{1}{t}\right) + \frac{1}{2}\frac{\sqrt{t^2-1}}{t^2} + C$$

5. First, do the partial fraction decomposition on the integrand:

$$\begin{aligned}
\frac{x-9}{x^2+3x-10} &= \frac{x-9}{(x+5)(x-2)} \text{ (Factor)} \\
&= \frac{A}{x+5} + \frac{B}{x-2} \text{ (Split, linear non repeated factors)} \\
x-9 &= A(x-2) + B(x+5) \text{ (Clear Fractions)} \\
\\
x-9 &= A(x-2) + B(x+5) \text{ (Set } x = 2 \text{ to determine } B) \\
2-9 &= A(2-2) + B(2+5) \\
-7 &= B(7) \\
B &= -1 \\
\\
x-9 &= A(x-2) + B(x+5) \text{ (Set } x = -5 \text{ to determine } A)
\end{aligned}$$

$$\begin{aligned}
-5-9 &= A(-5-2)+B(-5+5)\\
-14 &= A(-7)\\
A &= 2
\end{aligned}$$

$$\frac{x-9}{x^2+3x-10} = \frac{2}{x+5}-\frac{1}{x-2}$$

$$\int_3^5 \frac{x-9}{x^2+3x-10}\,dx = \int_3^5 \frac{2}{x+5}\,dx - \int_3^5 \frac{1}{x-2}\,dx$$

Substitute for first integral: $\begin{array}{ll} s=x+5 & \text{when } x=5\to s=10\\ ds=dx & \text{when } x=3\to s=8\end{array}$

Substitute for second integral: $\begin{array}{ll} t=x-2 & \text{when } x=5\to t=3\\ dt=dx & \text{when } x=3\to t=1\end{array}$

$$\begin{aligned}
&= 2\int_8^{10}\frac{ds}{s}-\int_1^3\frac{dt}{t}\\
&= 2\ln|s|_8^{10}-\ln|t|_1^3\\
&= 2\ln 10-2\ln 8-\ln 3+\ln 1\\
&= 2\ln 10-2\ln 8-\ln 3\\
&= \ln 100-\ln 64-\ln 3\\
&= \ln\left(\frac{100}{64\cdot 3}\right)\\
&= \ln\left(\frac{25}{48}\right)
\end{aligned}$$

6. First, do the partial fraction decomposition on the integrand:

$$\begin{aligned}
\frac{3x^2-2}{(x^2+1)(x+1)} &= \frac{3x^2-2}{(x^2+1)(x+1)} \text{ (Factor)}\\
&= \frac{Ax+B}{x^2+1}+\frac{C}{x+1} \text{ (Split)}\\
3x^2-2 &= (Ax+B)(x+1)+C(x^2+1) \text{ (Clear Fractions)}
\end{aligned}$$

$$\begin{aligned}
3(-1)^2-2 &= (Ax+B)(-1+1)+C((-1)^2+1) \text{ (Set } x=-1 \text{ to determine } C)\\
1 &= +C(2)\\
C &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
3x^2-2 &= (Ax+B)(x+1)+\frac{1}{2}(x^2+1)\\
3x^2-2-\frac{1}{2}(x^2+1) &= (Ax+B)(x+1) \text{ (Set } x=0 \text{ to determine } B)\\
3(0)^2-2-\frac{1}{2}(0^2+1) &= (A(0)+B)(0+1)\\
-\frac{5}{2} &= B
\end{aligned}$$

$$\begin{aligned}
3x^2-2-\frac{1}{2}(x^2+1) &= (Ax-\frac{5}{2})(x+1) \text{ (Set } x=1 \text{ to determine } A)\\
3(1)^2-2-\frac{1}{2}(1^2+1) &= (A-\frac{5}{2})(1+1)
\end{aligned}$$

$$
\begin{aligned}
A &= \frac{5}{2} \\
\frac{3x^2-2}{(x^2+1)(x+1)} &= \frac{5(x-1)}{2(x^2+1)} + \frac{1/2}{x+1} \\
&= \frac{5}{2}\left(\frac{x-1}{x^2+1}\right) + \left(\frac{1}{2}\frac{1}{x+1}\right) \\
&= \frac{5}{2}\left(\frac{x}{x^2+1}\right) - \frac{5}{2}\left(\frac{1}{x^2+1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right)
\end{aligned}
$$

So we have proven the statement. The first integral could be done using a simple u substitution ($u = x^2+1$), the second is an arctangent form, and the third is a logarithmic form.

7.

$$
\begin{aligned}
\int_0^\infty e^{-\pi x}\,dx &= \lim_{t\to\infty}\int_0^t e^{-\pi x}\,dx \text{ Substitution: } \begin{array}{ll} u=-\pi x, & x=0 \longrightarrow u=0 \\ du=-\pi\,dx, & x=t \longrightarrow u=-\pi t \end{array} \\
&= \lim_{t\to\infty}\frac{1}{(-\pi)}\int_0^{-\pi t} e^u\,du \\
&= -\frac{1}{\pi}\lim_{t\to\infty} e^u\Big|_0^{-\pi t} \\
&= -\frac{1}{\pi}\lim_{t\to\infty}\left(e^{-\pi t}-e^0\right) \\
&= -\frac{1}{\pi}\left(\lim_{t\to\infty} e^{-\pi t} - \lim_{t\to\infty} e^0\right) \\
&= -\frac{1}{\pi}(0-1) \\
&= \frac{1}{\pi}
\end{aligned}
$$