This is a set of practice test problems for Chapter 11. This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words. Make sure to also study any homework problems, problems done in class, and problems that are performed in the textbook.

There are sometimes more than one way to determine if a series converges or diverges. If you have an alternate solution than the one I have here and are unsure if it is correct, come and talk to me.

## Questions

**1.** The $n$th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = \dfrac{n-1}{n+1}$. Find $a_n$. Find $\sum a_n$.

**2.** Draw diagrams **and clearly explain** the Remainder Estimate for the Integral Test:

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx.$$

(Make sure you include the details of the integral test itself in your answer)

**3.** Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence using the limit comparison test.

**4.** Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ absolutely convergent, conditionally convergent, or divergent? Explain.

**5.** Test the series $\displaystyle\sum_{n=1}^{\infty} e^{-n} n!$ for convergence or divergence using the ratio test.

**6.** Find the **exact** sum of $\displaystyle\sum_{n=4}^{\infty} \frac{1}{(n-3)(n-1)}$ using partial fractions.

**7.** If the $n$th partial sum of a series $\sum a_n$ is given by $s_n = 3 - ne^{-n}$, find $\sum a_n$.

**8.** Show that series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{6n^2+4}$ diverges.

**9.** Is the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ absolutely convergent, conditionally convergent, or divergent?

## Solutions

**1.** The partial sums for the series are

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

and we also have

$$s_{n-1} = \sum_{i=1}^{n-1} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1}$$

Subtracting, we find an expression for $a_n$ in terms of the partial sums:

$$\begin{aligned}
a_n &= s_n - s_{n-1} \\
&= \frac{n-1}{n+1} - \frac{n-1-1}{n-1+1} \\
&= \frac{n-1}{n+1} - \frac{n-2}{n} \\
&= \frac{n^2 - n - n^2 + 2n - n + 2}{n(n+1)} \\
&= \frac{2}{n(n+1)} \\
\sum a_n &= \lim_{n\to\infty} s_n \\
&= \lim_{n\to\infty} \frac{n-1}{n+1} \\
&= \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} \\
&= 1
\end{aligned}$$

**2.** To use the remainder estimate for the integral test, the series $\sum a_n$ must be shown to converge by the integral test.

The integral test requires that we work with $f(x)$, where $f(n) = a_n$, and on the interval $[1, \infty)$, $f(x)$ is:
1) continuous,
2) positive,
3) decreasing.

For the series $\sum_{n=1}^{\infty} a_n$ to converge, $\int_1^{\infty} f(x)\,dx$ must converge.

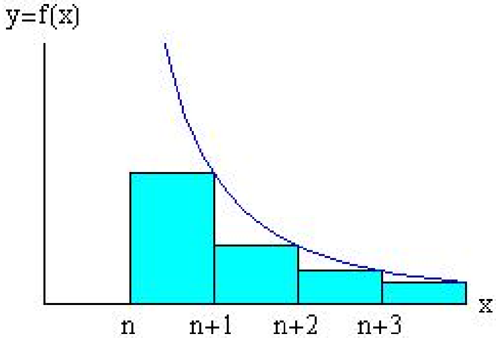
We approximate the sum $s = \sum_{i=1}^{\infty} a_i$ by the $n$th partial sum:

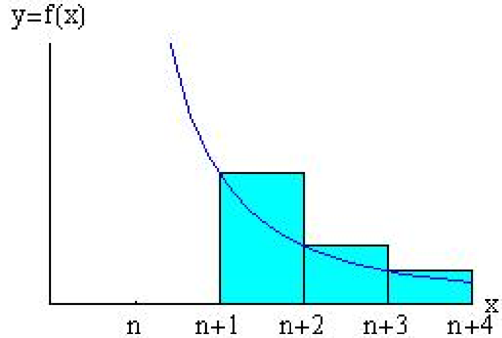
$$s_n = \sum_{i=1}^{n} a_i$$

which has a remainder of

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

Graphically, we can see how the remainder estimate for the integral test is obtained.




All the properties of $f(x)$ mentioned above are very important; without them, we would not know that the sketches we drew were an accurate reflection of the situation.

For the graph on the left: The area of all the rectangles is

$$a_{n+1} + a_{n+2} + a_{n+3} + \cdots = R_n$$

which is less than $\int_n^\infty f(x)\,dx$.

For the graph on the right: The area of all the rectangles is

$$a_{n+1} + a_{n+2} + a_{n+3} + \cdots = R_n$$

which is greater than $\int_{n+1}^\infty f(x)\,dx$.

Therefore, we have the remainder estimate for the integral test:

$$\int_{n+1}^\infty f(x)\,dx \le R_n \le \int_n^\infty f(x)\,dx.$$

**3.** Use the limit comparison test where

$$a_n = \frac{1}{2^n - 1} \text{ and } b_n = \frac{1}{2^n}$$

The series $\sum b_n$ is a convergent geometric series, since

$$\sum b_n = \sum \frac{1}{2^n} = \sum \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

which is a geometric series with $a = 1/2$, $r = 1/2$, $|r| < 1 \longrightarrow$ convergent.

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad = \quad \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} \\ &= \frac{1}{1-0} = 1 > 0 \end{aligned}$$

Since the limit of the ratio $a_n/b_n$ is greater than zero, and the series $\sum b_n$ converges, the series $\sum a_n$ converges by the limit comparison test.

**4.** Since the series we are asked to investigate is an alternating series, we should use the alternating series test.

$$a_n = \frac{(-1)^n}{n}$$

$$b_n = |a_n| = \frac{1}{n}$$

There are two conditions which must be satisfied for a series to be convergent by the alternating series test. They are:
1) $b_{n+1} < b_n$, and
2) $\lim_{n\to\infty} b_n = 0$.

In this case, we have:
$b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ so 1) is True, and
$\lim_{n\to\infty} \frac{1}{n} = 0 = \lim_{n\to\infty} b_n$, so 2) is True.

Therefore, $\sum a_n$ is convergent by the alternating series test.

To determine if the series $\sum a_n$ is conditionally convergent, we need to look at the series $\sum |a_n| = \sum b_n = \sum \frac{1}{n}$. This is a divergent $p$-series where $p = 1$.

So we have that $\sum a_n$ convergent and $\sum |a_n|$ divergent. This means that $\sum a_n$ is conditionally convergent.

**5.** Since there is a factorial in the series, we should try the ratio test. It can tell us if the series is absolutely convergent.

$$a_n = e^{-n} n!$$

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{e^{-(n+1)}(n+1)!}{e^{-n} n!} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)}{e} \right| \\ &= \frac{1}{e} \lim_{n\to\infty} |n+1| \\ &= \infty > 1 \end{aligned}$$

Therefore, $\sum a_n$ diverges by the ratio test.

**6.**

$$s_n = \sum_{i=4}^{n} \frac{1}{(i-3)(i-1)}$$

Partial fractions:

$$\begin{aligned}\frac{1}{(i-3)(i-1)} &= \frac{A}{i-3}+\frac{B}{i-1} \quad \text{split}\\ 1 &= A(i-1)+B(i-3) \quad \text{clear fractions}\end{aligned}$$

If we take $i = 1$, we get

$$1 = B(-2) \longrightarrow B = -\frac{1}{2}.$$

If we take $i = 3$, we get

$$1 = A(2) \longrightarrow A = \frac{1}{2}.$$

Therefore, we have

$$\frac{1}{(i-3)(i-1)} = \frac{1}{2}\left(\frac{1}{i-3}-\frac{1}{i-1}\right)$$

Now, we can simplify the partial sum

$$\begin{aligned}s_n &= \sum_{i=4}^{n}\frac{1}{(i-3)(i-1)}\\ &= \sum_{i=4}^{n}\left[\frac{1}{2}\left(\frac{1}{i-3}\right)-\frac{1}{2}\left(\frac{1}{i-1}\right)\right]\\ &= +\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n-4}+\frac{1}{n-3}\right)-\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}\right)\\ &= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n-2}-\frac{1}{n-1}\right)\\ &= \frac{3}{4}-\frac{1}{2n-4}-\frac{1}{2n-2}\end{aligned}$$

$$\sum_{n=4}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3}{4}-\frac{1}{2n-4}-\frac{1}{2n-2}\right) = \frac{3}{4}.$$

**7.** $\sum a_n = \sum_{i=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(3-ne^{-n}) \longrightarrow 3-\infty\cdot 0.$
Since this limit is an indeterminate product, to do this limit we need to compare with the continuous case and use L'Hospital's Rule.

Pick $f(x) = 3 - xe^{-x}$, so that $f(n) = a_n$. Now find the limit of the continuous function.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(3-xe^{-x})$$

$$\begin{aligned} &= 3 - \lim_{x\to\infty} \frac{x}{e^x} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient, so use L'HR.} \\ &= 3 - \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} \\ &= 3 - \lim_{x\to\infty} \frac{1}{e^x} \\ &= 3 - 0 = 3 \end{aligned}$$

Since $f(n) = a_n$, we can say that $\sum a_n = 3$.

**8.**

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^2}{6n^2+4} \\ &= \lim_{n\to\infty} \frac{1}{6+\frac{4}{n^2}} \\ &= \frac{1}{6+0} = \frac{1}{6} \neq 0 \end{aligned}$$

This diverges by the test for divergence.

**9.** We identify $a_n = (-1)^n \dfrac{\ln n}{n}$. Since this is an alternating series, we should try the alternating series test.

We identify $b_n = |a_n| = \dfrac{\ln n}{n}$.

For the first condition of the alternating series test, $b_{n+1} \leq b_n$ for all $n$, we need to work with the continuous function $f(x)$ where $f(n) = b_n$ and then show $f(x)$ is decreasing.

$$\begin{aligned} f(x) &= \frac{\ln x}{x} \\ f'(x) &= \frac{x\frac{1}{x} - \ln x(1)}{x^2} \\ &= \frac{1-\ln x}{x^2} \end{aligned}$$

So we have $f'(x) < 0$ if $1 - \ln x < 0$, which is the same as $x > e^1$. This means that for $n > e$, $b_{n+1} \leq b_n$.

I know $e < 3$, so let's just say that we have $b_{n+1} \leq b_n$ if $n \geq 3$.

The second condition of the alternating series test is to show $\lim_{n\to\infty} b_n = 0$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient}$$

We want to use L'Hospital's Rule to evaluate this limit, but we have to switch to a continuous function first, since the derivative is not defined for the discrete variable $n$. Pick $f(x)$ such that $f(n) = b_n$, and then proceed.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \longrightarrow \frac{\infty}{\infty} \quad \text{indeterminate quotient}$$

$$\begin{aligned} &= \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{(1)} \\ &= \lim_{x\to\infty} \frac{1}{x} \\ &= 0 \end{aligned}$$

Therefore, we also have $\lim_{n\to\infty} b_n = 0$.

The series $\sum_{n=3}^{\infty} a_n$ converges by the alternating series test.

Since $\sum_{n=3}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Now, we want to check the convergence of the series $\sum b_n = \sum |a_n|$, to determine if the series $\sum a_n$ is absolutely or conditionally convergent.

We can use the comparison test for this (or integral test). Here is how we construct our comparison series.

$$\begin{aligned} \ln n > 1 \quad &\text{if} \quad n > 3 \\ \frac{\ln n}{n} > \frac{1}{n} \quad &\text{if} \quad n > 3 \end{aligned}$$

So our comparison series should be $\sum c_n$ where $c_n = \frac{1}{n}$, which is the divergent $p$-series, with $p = 1$.

Since $b_n = \frac{\ln n}{n} > \frac{1}{n} = c_n$ if $n > 3$, and $\sum c_n$ diverges, we have that $\sum b_n$ must also diverge by the comparison test.

Therefore, $\sum a_n$ converges and $\sum |a_n| = \sum b_n$ diverges, so the series $\sum a_n$ is conditionally convergent.