This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. To prepare for the test:

- review homework,
- review WeBWorK,
- review examples from the text,
- review examples from class, and
- use the Study Guide.

The solutions are what I would accept on a test, but you may want to add more detail, and explain your steps with words.

You should know the Taylor series about $x = 0$ for common functions (see page 743 in text):

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \text{ geometric series} \\
&= 1 + x + x^2 + x^3 + \cdots \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty \\
&= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty \\
&= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1 \\
&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad |x| < 1 \\
&= 1 + kx + \frac{k(k-1)x^2}{2} + \frac{k(k-1)(k-2)x^3}{6} + \frac{k(k-1)(k-2)(k-3)x^4}{24} + \cdots
\end{aligned}$$

Definitions

A geometric series is $\dfrac{1}{1-y} =$

A power series is a series of the form:

- this is a power series about a,
- the c_n are the coefficients of the power series,
- the series may converge or diverge for each value of x.

The Taylor series of f about $x = a$ is:

- The radius of convergence of the Taylor series is R. It is often found using the Ratio Test.
- The interval of convergence of the Taylor series is the interval on which it converges.

The MacLaurin series of f is given by:

The Taylor polynomials of f about $x = a$ are given by:

The Remainder is given by:

Questions

1. Find the interval of convergence of (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(x+2)^n}{n2^n}$. (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{(n+1)^2}$.

2. Show that if $\lim_{n\to\infty}(|c_n|)^{1/n} = c$, then the radius of convergence of $\sum c_n x^n$ is $R = 1/c$.

3. Find the Maclaurin series for $f(x) = \cos x$ by using the definition of the Maclaurin series. Determine the radius of convergence. Use this to find $\displaystyle\lim_{x\to 0} \frac{1 - \frac{x^2}{2} - \cos x}{x^4}$.

4. Find the Taylor series for $f(x) = \sin x$ about $a = \pi/6$ by using the definition of the Taylor series. Do not determine the radius of convergence for this case (it can be done, but it is a bit unusual in how you do it).

5. Evaluate $\displaystyle\int \frac{1}{1+x^7}\,dx$ as a power series. Include a radius of convergence. Explain how you would then use this to approximate $\displaystyle\int_0^{1/2} \frac{1}{1+x^7}\,dx$ to 10^{-6} accuracy (explain the process, you can't work it out easily without a calculator).

6. The period a pendulum with length L that makes a maximum angle θ_0 with the vertical is given by physics as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - w^2 \sin^2 x}}$$

where $w = \sin(\theta_0/2)$.

Expand the integrand as a binomial series and show that if θ_0 is small then we can approximate $T \sim 2\pi\sqrt{L/g}$.

Solutions

1a. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}$.

Here, we use the ratio test to determine the radius of convergence first. $a_n = (-1)^n\dfrac{(x+2)^n}{n2^n}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(x+2)^{n+1}}{(n+1)2^{n+1}}\times(-1)^n\frac{n2^n}{(x+2)^n}\right| \\
&= \frac{1}{2}|x+2|\lim_{n\to\infty}\left|\frac{n}{(n+1)}\right| \\
&= \frac{1}{2}|x+2|\lim_{n\to\infty}\left|\frac{1}{(1+1/n)}\right| \\
&= \frac{1}{2}|x+2|
\end{aligned}$$

If this is less than 1, the series converges, so the series converges if $|x+2|<2$.
This is the same as the interval $-4<x<0$, since the center is $a=-2$ and the radius of convergence is $R=2$.

We need to check endpoints individually, since the ratio test tells us nothing there.

<u>Consider $x=-4$</u>: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-4+2)^n}{n2^n} = \sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$

This is the divergent harmonic series.

<u>Consider $x=0$</u>: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(+2)^n}{n2^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$

This is an alternating series, with $b_n=\frac{1}{n}$. It is convergent since $b_{n+1}=\frac{1}{n+1}<\frac{1}{n}=b_n$ and $\lim_{n\to\infty}b_n=0$.

So the interval of convergence for $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}$ is $-4<x\le 0$.

1b. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{3^nx^n}{(n+1)^2}$.

Here, we use the ratio test to determine the radius of convergence first. $a_n=\dfrac{3^nx^n}{(n+1)^2}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{(n+2)^2}\times\frac{(n+1)^2}{3^nx^n}\right| \\
&= 3|x|\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+2)^2}\right| \\
&= 3|x|\lim_{n\to\infty}\left|\frac{(1+1/n)^2}{(1+2/n)^2}\right| \\
&= 3|x|
\end{aligned}$$

If this is less than 1, the series converges, so the series converges if $|x| < \frac{1}{3}$.
This is the same as the interval $-1/3 < x < 1/3$, since the center is $a = 0$ and the radius of convergence is $R = 1/3$.

We need to check endpoints individually, since the ratio test tells us nothing there.

Consider $x = 1/3$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(1/3)^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$

This is a convergent p-series, with $p = 2$.

Consider $x = -1/3$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(-1/3)^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}$

This is an alternating series, with $b_n = \frac{1}{(n+1)^2}$. It is convergent since $b_{n+1} = \frac{1}{(n+2)^2} < \frac{1}{(n+1)^2} = b_n$ and $\lim_{n\to\infty} b_n = 0$.

So the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{(n+1)^2}$ is $-1/3 \le x \le 1/3$.

2. Show that if $\lim_{n\to\infty}(|c_n|)^{1/n} = c$, then the radius of convergence of $\sum c_n x^n$ is $R = 1/c$.

The first limit looks kind of like the root test. We have used the ratio test almost exclusively to find the radius of convergence, but the root test may also work in some cases. The ratio test is definitely the way to go when computing Taylor series due to the factorials that show up!

The root test for $\sum c_n x^n$ would give us: $a_n = c_n x^n$, and

$$\begin{aligned} \lim_{n\to\infty}(|a_n|)^{1/n} &= \lim_{n\to\infty}(|c_n x^n|)^{1/n} \\ &= |x| \lim_{n\to\infty}(|c_n|)^{1/n} \\ &= |x|c \text{ (using the result given to us)} \end{aligned}$$

If this is less than 1, the series converges, so the series converges if $|x| < \frac{1}{c}$.

3. Find the Maclaurin series for $f(x) = \cos x$ by using the definition of the Maclaurin series. Determine the radius of convergence. Use this to find $\displaystyle\lim_{x\to 0} \frac{1 - \frac{x^2}{2} - \cos x}{x^4}$.

Let's construct a table which will give us the derivatives, and enable us to calculate $f^{(n)}(a)$. We will want the general form, so we should try and write things in ways in which the pattern becomes evident.

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
5	$-\sin x$	0

To see the general form, it is best to expand the Taylor series out and then try to recognize the pattern.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

So we can see that the general form is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

Now we want to find the radius of convergence, R. We can do this using the ratio test, where $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right| \\
&= |x^2| \lim_{n\to\infty}\left|\frac{(2n)!}{(2n+2)!}\right| \\
&= |x^2| \lim_{n\to\infty}\left|\frac{1}{(2n+1)(2n+2)}\right| \\
&= 0
\end{aligned}$$

So the series is absolutely convergent for all values of x, $R = \infty$.

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\frac{x^2}{2}-\cos x}{x^4} &= \lim_{x\to 0}\frac{1-\frac{x^2}{2}-(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots)}{x^4} \\
&= \lim_{x\to 0}\frac{-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\cdots}{x^4} \\
&= \lim_{x\to 0} -\frac{1}{4!}+\frac{x^2}{6!}-\frac{x^4}{8!}+\cdots \\
&= -\frac{1}{4!}+0-0+\cdots = -\frac{1}{24}
\end{aligned}$$

4. Find the Taylor series for $f(x) = \sin x$ about $a = \pi/6$ by using the definition of the Taylor series. Determine the radius of convergence.

This one is about keeping track of patterns, and seeing what happens to sine when you expand about something other than $x = 0$.

The table we get is:

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(\pi/6)$
0	$\sin x$	$+\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$+\frac{1}{2}$
5	$\cos x$	$\frac{\sqrt{3}}{2}$
6	$-\sin x$	$-\frac{1}{2}$
7	$-\cos x$	$-\frac{\sqrt{3}}{2}$
8	$\sin x$	$+\frac{1}{2}$

The table repeats itself for with a period of 4, the number of derivatives it takes to get back to $\sin x$.

So the pattern comes in two parts, one for odd powers of x and another for even powers of x.

If the power is even, $n = 2k$, $f^{(2k)}(a) = (-1)^k \frac{1}{2}$. Odd, $n = 2k+1$, $f^{(2k+1)}(a) = (-1)^k \frac{\sqrt{3}}{2}$.

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= \sum_{k=0}^{\infty} \frac{f^{(2k)}(a)}{(2k)!}(x-a)^{2k} + \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(a)}{(2k+1)!}(x-a)^{2k+1} \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x-\frac{\pi}{6}\right)^{2k} + \frac{\sqrt{3}}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(x-\frac{\pi}{6}\right)^{2k+1}
\end{aligned}$$

5. Evaluate $\int \frac{1}{1+x^7}dx$ as a power series. Include a radius of convergence. Use this to approximate $\int_0^{1/2} \frac{1}{1+x^7}dx$ correct to 10^{-6}. Explain why this procedure is valid, and how you know your solution is accurate.

The integrand is a geometric series. Let's use that to our advantage.

$$\begin{aligned}
\frac{1}{1+x^7} &= \frac{1}{1-(-x^7)} \\
&= \sum_{n=0}^{\infty}(-x^7)^n, |-x^7| < 1 \\
&= \sum_{n=0}^{\infty}(-1)^n x^{7n}, |x| < 1 \\
\int \frac{1}{1+x^7}\,dx &= \int \sum_{n=0}^{\infty}(-1)^n x^{7n}\,dx, |x| < 1 \\
&= \sum_{n=0}^{\infty}(-1)^n \int x^{7n}\,dx, |x| < 1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1}x^{7n+1} + C, |x| < 1
\end{aligned}$$

Since $x \in (0, 1/2)$ is inside the interval of convergence, we can use this formula to evaluate the integral from $x = 0$ to $x = 1/2$.

$$\begin{aligned}\int_0^{1/2} \frac{1}{1+x^7}\,dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1} x^{7n+1}\Big|_0^{1/2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{7n+1} \left(\frac{1}{2}\right)^{7n+1}\end{aligned}$$

Since the series is an alternating series, we can use the alternating series remainder estimate to know when we have the answer accurate to 10^{-6}. We just need to keep n terms and once $b_{n+1} < 10^{-6}$ we stop.

6. The period a pendulum with length L that makes a maximum angle θ_0 with the vertical is given by physics as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - w^2 \sin^2 x}}$$

where $w = \sin(\theta_0/2)$.

Expand the integrand as a binomial series and show that if θ_0 is small then we can approximate $T \sim 2\pi\sqrt{L/g}$.

$$\begin{aligned}T &= 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - w^2 \sin^2 x}} \\ &= 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1 - w^2 \sin^2 x\right)^{-1/2} dx\end{aligned}$$

binomial series with $k = -1/2$ and $y = -w^2 \sin^2 x$, keep only first two terms and see what happens.

$$\begin{aligned}
T &= 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} (1+y)^k \, dx \\
&\sim 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} (1+ky+\cdots) \, dx \\
&\sim 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1+\frac{1}{2}w^2\sin^2(x)\right) dx \\
&\sim 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1+\frac{1}{2}w^2\sin^2(x)\right) dx \\
&\sim 4\sqrt{\frac{L}{g}} \left(\int_0^{\pi/2} dx + \frac{1}{2}w^2 \int_0^{\pi/2} \sin^2(x)\, dx\right) \\
&\sim 4\sqrt{\frac{L}{g}} \left(\frac{\pi}{2} + \frac{1}{4}w^2 \int_0^{\pi/2} (1-\cos(2x)\, dx\right) \\
&\sim 4\sqrt{\frac{L}{g}} \left(\frac{\pi}{2} + \frac{1}{4}w^2 \left(x - \frac{1}{2}\sin(2x)\right)_0^{\pi/2}\right) \\
&\sim 4\sqrt{\frac{L}{g}} \left(\frac{\pi}{2} + \frac{1}{4}w^2\frac{\pi}{2}\right) \\
&\sim 2\pi\sqrt{\frac{L}{g}} \left(1 + \frac{1}{4}w^2\right)
\end{aligned}$$

If θ_0 is small, then $1 \gg \frac{1}{4}w^2$ since $w = \sin(\theta_0/2) \sim \sin 0 = 0$.

Therefore,

$$T \sim 2\pi\sqrt{\frac{L}{g}} \text{ if } \theta_0 \text{ is small.}$$