This is in **no way** an inclusive set of problems–there can be other types of problems on the actual test. To prepare for the test:

- review homework,
- review WeBWorK,
- review examples from the text,
- review examples from class, and
- use the Study Guide.

Make sure you are comfortable with:

- basic integrals and techniques of integration like
  - substitution
  - trig integrals
  - parts
  - trig substitution
  - partial fractions
- trig identities (for the integrals that have to be done),
- the rules for working with exponential/logarithmic equations (which are important in population models),
- direction fields and how they are created; how equilibrium solutions appear in direction fields,
- how Euler's method is derived and how it relates to direction fields,
- solving differential equations (DE) and initial value problems (IVP) via
  - separable equations technique $\dfrac{dy}{dt} = f(t)g(y)$.
  - predator-prey (equilibrium solutions, solving system by taking ratio of derivatives)
- how to get explicit solution to Logistic (done in text) as well as population models we did in class.
- how to find orthogonal trajectories
- parametric functions
- arclength
- surface area

Useful Information which will be provided:

$$\begin{aligned}\cos^2 x + \sin^2 x &= 1\\ \cos(x+y) &= \cos x \cos y - \sin x \sin y\\ \sin(x+y) &= \sin x \cos y + \cos x \sin y\\ (ds)^2 &= (dx)^2 + (dy)^2\end{aligned}$$

Questions

1. Find an explicit solution to the initial value problem $\frac{dP}{dt} = kP\cos^2(rt-\phi)$, $P(0) - P_0$, where k, r, and ϕ are unspecified constants, and P represents a population of animals.

2. Derive Euler's method which is used to numerically solve the initial value problem

$$y' = f(x, y), y(x_0) = y_0,$$

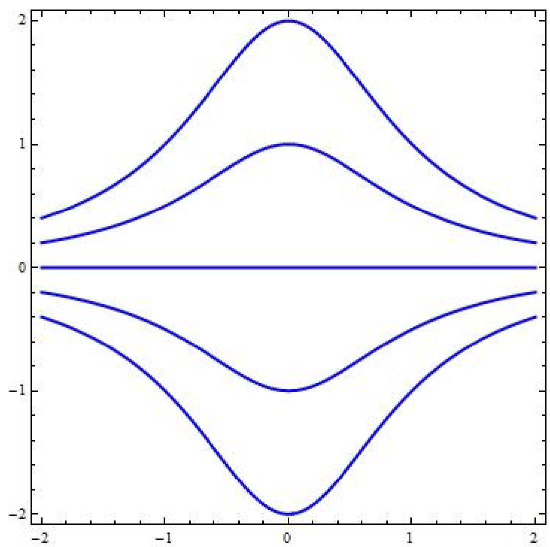
using the definition of derivative and an appropriate approximation.

3. Find the orthogonal trajectories of the family of curves $y = (x + k)^{-1}$.

4. Find the slope of the line segments of the direction field at the points $(0, 0)$ and $(1, 1)$ for the differential equation $y' = x^2 + y^2$. Describe what these lines tell you about the behaviour of the solution at these two points.

5. A young scientist starts a bacteria culture, and then gets called away on urgent business (lunch). She realizes at lunch that she forgot to measure the size of her bacteria culture, but knows she can estimate it accurately using mathematics, since the bacteria growth will be proportional to the population size in the early stages. After lunch, which was 57 minutes long, the scientist measures the size of the culture and determines it to be 435 bacteria. She waits one hour, and then measures the population size, finding it to be 9897 bacteria. How many bacteria did her initial culture contain?

6. Given the family of curves $y = \dfrac{k}{1 + x^2}$, which are given in a sketch by



(a.) Sketch what the orthogonal trajectories to the family of curves would look like.

(b.) Show the family satisfies the differential equation

$$\frac{dy}{dx} = \frac{-2xy}{1 + x^2}.$$

(c.) Show that orthogonal trajectories to this family are

$$\ln|x| + \frac{x^2}{2} = y^2 + C.$$

7. Find the length of the curve $y = \ln(\cos x), \quad 0 \le x \le \pi/3$.

8. Find the surface area of the surface obtained by rotating $y = x^2$, $0 \le x \le 1$, about the y-axis.

Solutions

1. This is separable so we will just need the use of our integration skills.

$$\begin{aligned}
\frac{dP}{dt} &= kP\cos^2(rt-\phi) \\
\frac{dP}{P} &= k\cos^2(rt-\phi)\,dt \\
\int\frac{dP}{P} &= k\int\cos^2(rt-\phi)\,dt \\
& \text{Substitution: } u = rt-\phi, \quad du = r\,dt \\
\ln|P| &= \frac{k}{r}\int\cos^2 u\,du \\
\ln|P| &= \frac{k}{r}\int\frac{1}{2}(1+\cos(2u))\,du \\
\ln|P| &= \frac{k}{2r}\int du + \frac{k}{2r}\int\cos(2u)\,du \\
\ln|P| &= \frac{k}{2r}u + \frac{k}{4r}\sin(2u) + c \\
\ln|P| &= \frac{k}{2r}(rt-\phi) + \frac{k}{4r}\sin(2rt-2\phi) + c
\end{aligned}$$

This is an implicit solution to the differential equation, and represents a family of curves. We can use our initial condition now to determine the value of c.

$$\begin{aligned}
\ln|P_0| &= \frac{k}{2r}(-\phi) + \frac{k}{4r}\sin(-2\phi) + c \\
c &= +\frac{k\phi}{2r} - \frac{k}{4r}\sin(-2\phi) + \ln|P_0| \\
&= \frac{k\phi}{2r} + \frac{k}{4r}\sin(2\phi) + \ln|P_0|
\end{aligned}$$

So an implicit solution to the initial value problem is

$$\ln|P| = \frac{k}{2r}(rt-\phi) + \frac{k}{4r}\sin(2rt-2\phi) + \frac{k\phi}{2r} + \frac{k}{4r}\sin(2\phi) + \ln|P_0|$$

Since we want an explicit solution, we need to solve the above equation for P.

$$\begin{aligned}
|P| &= \exp\left(\frac{k}{2r}(rt-\phi) + \frac{k}{4r}\sin(2rt-2\phi) + \frac{k\phi}{2r} + \frac{k}{4r}\sin(2\phi) + \ln|P_0|\right) \\
|P| &= |P_0|\exp\left(\frac{k}{2r}(rt-\phi) + \frac{k}{4r}\sin(2rt-2\phi) + \frac{k\phi}{2r} + \frac{k}{4r}\sin(2\phi)\right) \\
P(t) &= P_0\exp\left(\frac{k}{2r}(rt-\phi) + \frac{k}{4r}\sin(2rt-2\phi) + \frac{k\phi}{2r} + \frac{k}{4r}\sin(2\phi)\right) \\
&= P_0\exp\left(\frac{kt}{2} + \frac{k}{4r}\sin(2rt-2\phi) + \frac{k}{4r}\sin(2\phi)\right)
\end{aligned}$$

where we dropped the absolute values since $P(t) > 0$ for populations.

2. The differential equation is

$$\frac{dy}{dx} = f(x,y) = f(x,y(x)).$$

The derivative is defined as

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{y(x+h)-y(x)}{h}$$

So we can approximate the derivative by taking h as small, but not zero.

$$\frac{dy}{dx} \sim \frac{y(x+h)-y(x)}{h}$$

The differential equation is approximated by

$$\frac{dy}{dx} = f(x,y(x)) \quad \text{(The original differential equation)}$$
$$\frac{y(x+h)-y(x)}{h} = f(x,y(x)) \quad \text{(The numerical approximation)}$$
$$y(x+h) = hf(x,y(x)) + y(x) \quad \text{(Rewrite as an iteration equation)}$$

Therefore, if we know the solution at x, we can get the solution at $x+h$ using this equation. We start at $x = x_0$ (we will need an initial condition to get started), and step through the solution, moving to $x+h$, then $x+2h$, then $x+3h$, etc. for as long as we like. There will be an error in each step, and if the step size h is too large this numerical approximation will not be a good representation of the actual solution.

The iteration equation is typically written in the following form:

$$\begin{aligned} x_n &= x_{n-1} + h \\ y_n &= y_{n-1} + hf(x_{n-1}, y_{n-1}) \end{aligned}$$

and we initialize by specifying x_0 and y_0 and let $n = 1, 2, 3, \ldots$ to get the solution.

3. First, we construct the differential equation the original family of curves satisfies. To construct the differential equation, we take the derivative of the original family and eliminate the parameter k:

$$\begin{aligned} y &= (x+k)^{-1} \\ \frac{d}{dx}(y &= (x+k)^{-1}) \\ \frac{dy}{dx} &= \frac{d}{dx}(x+k)^{-1} \\ \frac{dy}{dx} &= -(x+k)^{-2} \end{aligned}$$

We now eliminate the k using the original equation:

$$y = (x+k)^{-1} \longrightarrow k = \frac{1}{y} - x$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= -(x+\frac{1}{y}-x)^{-2}\\ \frac{dy}{dx} &= -y^2\end{aligned}$$

The differential equation which the family of curves $y=(x+k)^{-1}$ satisfies is $\dfrac{dy}{dx}=-y^2$.

The differential equation which the orthogonal trajectories will satisfy is therefore $-\dfrac{dx}{dy}=-y^2$,

(this is because the slope of the tangent lines for the orthogonal trajectories should be negative reciprocals of the slope of the original family of curves).

Now, we solve this differential equation to get the family of orthogonal trajectories.

$$\begin{aligned}-\frac{dx}{dy} &= -y^2\\ dx &= y^2\,dy \quad \text{(separate)}\\ \int dx &= \int y^2\,dy \quad \text{(integrate)}\\ x+c_1 &= \frac{y^3}{3}+c_2 \quad (c_1, c_2 \text{ are constants of integration})\\ x &= \frac{y^3}{3}+c \quad (c=c_2-c_1)\end{aligned}$$

The orthogonal trajectories of $y=(x+k)^{-1}$ are given (implicitly) by the equation $x=\dfrac{y^3}{3}+c$.

4. This question is asking us to think about direction fields, although it doesn't say that specifically. The slope of the curve at $(0,0)$ is given by the derivative of the function at the point $(0,0)$. Therefore, the slope is just

$$\left.\frac{dy}{dx}\right|_{(0,0)} = f(0,0) = (0)^2+(0)^2 = 0.$$

Therefore, since slope is the rise over the run, we have

$$\text{slope } = \frac{\text{rise}}{\text{run}} = 0,$$

which means we have a horizontal line. The solution may have a local maximum or minimum at the point $(0,0)$.

At $(1,1)$, we have

$$\left.\frac{dy}{dx}\right|_{(1,1)} = f(1,1) = (1)^2+(1)^2 = 2.$$

Therefore, since slope is the rise over the run, we have

$$\text{slope } = \frac{\text{rise}}{\text{run}} = \frac{2}{1},$$

and the solution is increasing (since the derivative is greater than zero) at the point $(1,1)$.

5. Since the bacteria culture will grow at a rate proportional to the population size in the early stages of its growth, the population y will satisfy the differential equation

$$\frac{dy}{dt} = ky.$$

We can easily solve this, since it is separable.

$$\begin{aligned}
\frac{dy}{dt} &= ky \\
\frac{dy}{y} &= k\,dt \\
\int \frac{dy}{y} &= \int k\,dt \\
\ln|y| + c_1 &= kt + c_2 \\
\ln|y| &= kt + c_3 \\
|y| &= e^{kt+c_3} = e^{kt}e^{c_3} \\
y &= \pm e^{c_3}e^{kt} \\
y &= Ae^{kt}
\end{aligned}$$

We can use the two measurements the scientist made to determine the constants A and k. The second measurement occurred 60 minutes after the first, or 117 minutes after the bacteria culture was first created. The data points we have are $(57, 435)$ and $(117, 9897)$. The system of equations we need to solve is

$$y(57) = 435 = Ae^{57k} \quad (1)$$
$$y(117) = 9897 = Ae^{117k}. \quad (2)$$

Dividing the two equations above ((1)/(2)), we get

$$\begin{aligned}
\frac{435}{9897} &= \frac{e^{57k}}{e^{117k}} = e^{(57-117)k} \\
\frac{435}{9897} &= e^{-60k} \\
\ln\left(\frac{435}{9897}\right) &= -60k \\
k &= -\frac{1}{60}\ln\left(\frac{435}{9897}\right)
\end{aligned}$$

Sub this back into Eq. (1) to get A:

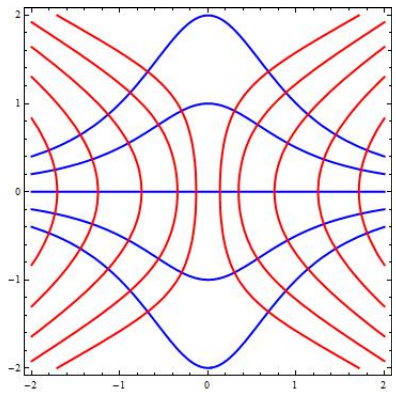
$$\begin{aligned}
435 &= Ae^{-\frac{57}{60}\ln\left(\frac{435}{9897}\right)} \\
A &= 435e^{\frac{57}{60}\ln\left(\frac{435}{9897}\right)}
\end{aligned}$$

The population after t minutes is given by

$$y(t) = 435 \exp\left(\frac{57}{60} \ln\left(\frac{435}{9897}\right)\right) \exp\left(-\frac{1}{60} \ln\left(\frac{435}{9897}\right) t\right).$$

The initial population of bacteria is $435 \exp\left(\frac{57}{60} \ln\left(\frac{435}{9897}\right)\right)$.

6. (a)



The differential equation satisfied by the original family of curves is found by differentiating:

$$\begin{aligned} y &= \frac{k}{1+x^2} \\ \frac{d}{dx}(y &= \frac{k}{1+x^2}) \\ \frac{dy}{dx} &= k\frac{d}{dx}\left(\frac{1}{1+x^2}\right) \\ &= -k(1+x^2)^{-2}(2x) \\ &= -2xk\frac{1}{(1+x^2)^2} \\ &= -2x(y(1+x^2))\frac{1}{(1+x^2)^2} \\ \frac{dy}{dx} &= -\frac{2xy}{(1+x^2)} \end{aligned}$$

The differential equation satisfied by the orthogonal trajectories is therefore given by (derivatives must be negative reciprocals):

$$\begin{aligned}
-\frac{dx}{dy} &= -\frac{2xy}{(1+x^2)} \\
\frac{(1+x^2)}{x}\,dx &= 2y\,dy \quad \text{(separate)} \\
\int \frac{(1+x^2)}{x}\,dx &= \int 2y\,dy \quad \text{(integrate)} \\
\int \frac{1}{x}\,dx + \int x\,dx &= \int 2y\,dy \\
\ln|x| + \frac{x^2}{2} + c_1 &= y^2 + c_2 \\
\ln|x| + \frac{x^2}{2} &= y^2 + c \quad (c = c_2 - c_1)
\end{aligned}$$

The orthogonal trajectories are given implicitly by the equation $\ln|x| + \frac{x^2}{2} = y^2 + c$, where c is a constant.

7. Arc length!

$$\begin{aligned}
y &= \ln(\cos x) \\
\frac{dy}{dx} &= -\frac{\sin x}{\cos x} = -\tan x \\
ds &= \sqrt{(dx)^2 + (dy)^2} \\
&= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\
&= \sqrt{1 + (\tan x)^2}\,dx \\
&= \sec x\,dx \\
s &= \int_0^{\pi/3} \sec x\,dx \\
&= \ln\left|\sec x + \tan x\right|_0^{\pi/3} \\
&= \ln\left|\sec(\pi/3) + \tan(\pi/3)\right| - \ln\left|\sec(0) + \tan(0)\right| \\
&= \ln\left|2 + \sqrt{3}\right| - \ln\left|(1) + (0)\right| \\
&= \ln\left|2 + \sqrt{3}\right|
\end{aligned}$$

8. Surface area! You should include a diagram to motivate where the formula comes from.

$$\begin{aligned}
y &= x^2 \\
\frac{dy}{dx} &= 2x \\
ds &= \sqrt{(dx)^2 + (dy)^2} \\
&= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \sqrt{1 + 4x^2}\, dx \\
\text{Surface Area } &= \int_0^1 2\pi x\, ds \\
&= 2\pi \int_0^1 x\sqrt{1+4x^2}\, dx \qquad \text{let } u = 1 + 4x^2,\ du = 8x\, dx,\ x = 0 \to u = 1, x = 1 \to u = 5 \\
&= \frac{\pi}{4} \int_1^5 \sqrt{u}\, du \\
&= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^5 \\
&= \frac{\pi}{6} \left(5\sqrt{5} - 1\right)
\end{aligned}$$