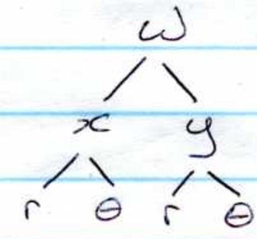
14.4.42

$\omega = f(x,y)$

$x = r\cos\theta$

$y = r\sin\theta$



$$\frac{\partial \omega}{\partial r} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{\partial \omega}{\partial r} = f_x \cos\theta + f_y \sin\theta \qquad (1)$$

$$\frac{\partial \omega}{\partial \theta} = \frac{\partial \omega}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \omega}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= f_x(-r\sin\theta) + f_y(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial \omega}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta \qquad (2)$$

Use Cramer's Rule to solve (1) & (2) for f_x, f_y:

$$f_x = \frac{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}}{\begin{vmatrix} \partial\omega/\partial r & \sin\theta \\ \frac{1}{r}\partial\omega/\partial\theta & \cos\theta \end{vmatrix}}$$

flip!!

$$= \frac{\partial \omega}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \omega}{\partial \theta}\sin\theta$$

$$f_y = \frac{\begin{vmatrix} \cos\theta & \partial\omega/\partial r \\ -\sin\theta & \frac{1}{r}\partial\omega/\partial\theta \end{vmatrix}}{1} = \frac{1}{r}\frac{\partial \omega}{\partial \theta}\cos\theta + \frac{\partial \omega}{\partial r}\sin\theta$$

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial \omega}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \omega}{\partial \theta}\right)^2$$ (cross terms cancel!)

14.4.45 helix $x = \cos t$

$y = \sin t$

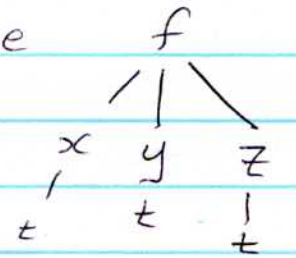
$z = t$

$f(x,y,z)$ has the following properties:

$$f_x = \cos t$$
$$f_y = \sin t$$
$$f_z = t^2 + t - 2.$$

f has extreme values (potentially) if $\frac{df}{dt} = 0$.

use



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (\cos t)(-\sin t) + (\sin t)(\cos t) + (1)(t^2+t-2)$$

$$0 = t^2 + t - 2$$

$$t = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2, 1.$$

$\frac{dx}{dt} = -\sin t$

$\frac{dy}{dt} = \cos t$

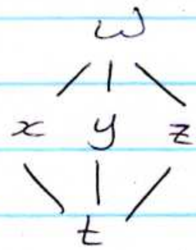
$\frac{dz}{dt} = 1$

f takes on extreme values at the following points along the helix:

$$(\cos 1, \sin 1, 1)$$
$$(\cos(-2), \sin(-2), -2).$$

14.4.46

$\omega = x^2 e^{2y} \cos 3z$



$x = \cos t$

$y = \ln(t+2)$

$z = t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \left(2x\, e^{2y} \cos 3z\right)\left(-\sin t\right) + \frac{\left(2x^2 e^{2y} \cos 3z\right)}{t+2} + (1)\left(-3x^2 e^{2y} \sin 3z\right)$$

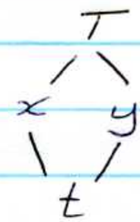
evaluate at $x = 1$, $y = \ln 2$, $z = 0$ $\Rightarrow t = 0$

$$\frac{dw}{dt} = 0 + \frac{8}{2} - 0 = 4.$$

14.4.48

$x = 2\sqrt{2}\cos t \qquad 0 \le t \le 2\pi$

$y = \sqrt{2}\sin t$



$\frac{\partial T}{\partial x} = y \qquad \frac{\partial T}{\partial y} = x.$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$= y\left(-2\sqrt{2}\sin t\right) + x\left(\sqrt{2}\cos t\right)$$

$$= -4\sin^2 t + 4\cos^2 t$$

$\frac{dT}{dt} = 0 \Rightarrow \frac{\sin^2 t}{\cos^2 t} = 1 \Rightarrow \tan t = 1 \Rightarrow t = \pi/4.$

Does $t = \pi/4$ give max or min? $\Rightarrow$ Second derivative test.

$$\frac{d^2T}{dt^2} = \frac{d}{dt}\left(-4\sin^2 t + 4\cos^2 t\right)$$

$$= -8\sin t\cos t - 8\cos t\sin t$$

$$= -16\sin t\cos t$$

Since $\underbrace{\frac{d^2T}{dt^2}}_{=-8} < 0$ at $t = \pi/4$, T is concave down at $t = \pi/4$

$\Rightarrow$ T has max at $t = \pi/4$.

Another solution is $t = \pi/4 + \pi = 5\pi/4$ (since $\tan t$ has period π).

$\left.\frac{d^2T}{dt^2}\right|_{t=5\pi/4} = -8 < 0$. T also has a max at $t = 5\pi/4$.

There is also a solution at $t = 3\pi/4$.

$\left.\frac{d^2T}{dt^2}\right|_{t=3\pi/4} = 8 > 0$. T has a min at $t = 3\pi/4$.

We also have a min at $t = 7\pi/4$.

14.4.48 continued

The max & min values if $T = xy - 2$ are

$$\text{max: } T = 0 \qquad \text{min } T = -4.$$

See MMA file for a sketch.