$\boxed{15.4.14}$

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\, dx\, dy$$

$$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x+y)\, dx\, dy$$

$$= \int_0^2 \left( x^2 + xy \right)\Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= \int_0^2 \left( \cancel{4-y^2} + \sqrt{4-y^2}\, y \cancel{- 4 + y^2} + \sqrt{4-y^2}\, y \right) dy$$

$$= 2 \int_0^2 \sqrt{4-y^2}\, y\, dy \qquad \text{let } \omega = 4-y^2$$

$$d\omega = -2y\,dy$$

when $y=0$, $\omega=4$

$y=2$, $\omega=0$

$$= 2 \int_4^0 \sqrt{\omega}\, \frac{d\omega}{(-2)}$$

$$= \int_0^4 \omega^{1/2}\, d\omega$$

$$= \frac{\omega^{3/2}}{3/2} \Big|_0^4$$

$$= \frac{2}{3} (4)^{3/2} = \frac{2}{3}(8) = \frac{16}{3}$$

15.4.16

$$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x^2} xz \Big|_{z=3}^{z=4-x^2-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x^2} \left( x(4-x^2-y) - 3x \right) dy\, dx$$

$$= \int_0^1 \int_0^{1-x^2} \left( x - x^2 - yx \right) dy\, dx$$

$$= \int_0^1 \left( xy - x^2y - \frac{y^2x}{2} \right)\Big|_0^{1-x^2} dx$$

$$= \int_0^1 \left( x(1-x^2) - x^2(1-x^2) - \frac{(1-x^2)^2 x}{2} \right) dx$$

$$= \int_0^1 \left( x - \cancel{x^3} - x^2 + \cancel{x^3} - \frac{x}{2} - \frac{x^5}{2} + x^3 \right) dx$$

$$\cancel{= \int_0^1 \left( x - \frac{1}{2} - \frac{x^4}{2} \right) dx} \qquad = \int_0^1 \left( \frac{x}{2} - x^2 - \frac{x^5}{2} + x^3 \right) dx$$

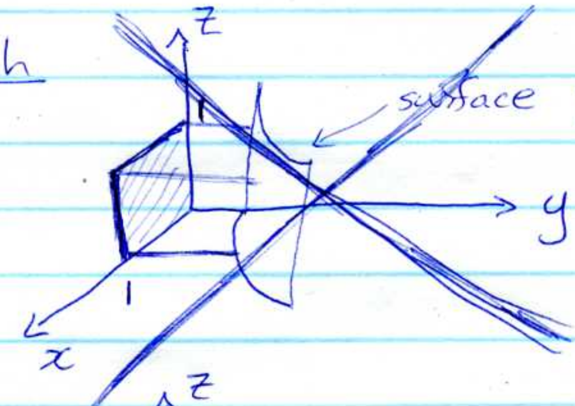
$$\cancel{= \frac{x^2}{2} - \frac{1}{2}x - \frac{x^5}{10} \Big|_0^1} \qquad = \frac{x^2}{4} - \frac{x^3}{3} - \frac{x^6}{12} + \frac{x^4}{4} \Big|_0^1$$

$$\cancel{= \frac{1}{2} - \frac{1}{2} - \frac{1}{10} = -\frac{1}{10}} \qquad = \frac{1}{12}$$

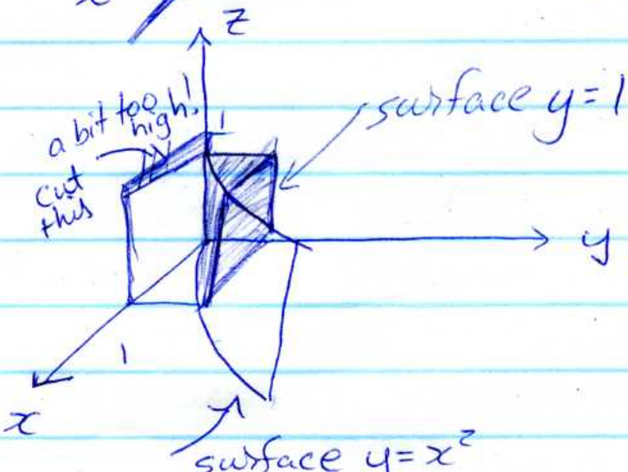
$\boxed{15.4.42}$ $\displaystyle\int_0^1\int_0^1\int_{x^2}^1 12xze^{zy^2}\,dy\,dx\,dz = \underline{I}$ Problem: can't do y integral.

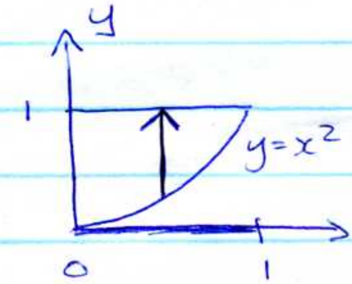
$R = \{(x,y,z) \mid 0 \le z \le 1,\ 0 \le x \le 1,\ x^2 \le y \le 1\}$

sketch




Whoops! bad sketch. $y=x^2$ should go through origin.

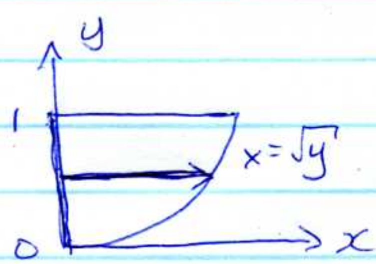



$R = \{(x,y,z) \mid 0 \le x \le 1,\ x^2 \le y \le 1,\ 0 \le z \le 1\}$.

This is the description of original integral.

Try




$R = \{(x,y,z) \mid 0 \le y \le 1,\ 0 \le x \le \sqrt{y},\ 0 \le z \le 1\}$

Let's see if this allows us to do the integral by hand.

15.4.42 continued

$$I = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2}\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^1 6x^2ze^{zy^2}\Big|_{x=0}^{x=\sqrt{y}}\,dy\,dz$$

$$= \int_0^1 \int_0^1 6yze^{zy^2}\,dy\,dz$$

$w = y^2$

$dw = 2y\,dy$

when $y=0$ $w=0$

$y=1$ $w=1$

$$= \int_0^1 \int_0^1 3ze^{zw}\,dw\,dz$$

$$= \int_0^1 3e^{zw}\Big|_{\cancel{z}w=0}^{w=1}\,dz$$

$$= \int_0^1 (3e^z - 3)\,dz$$

$$= 3e^z - 3z\Big|_0^1$$

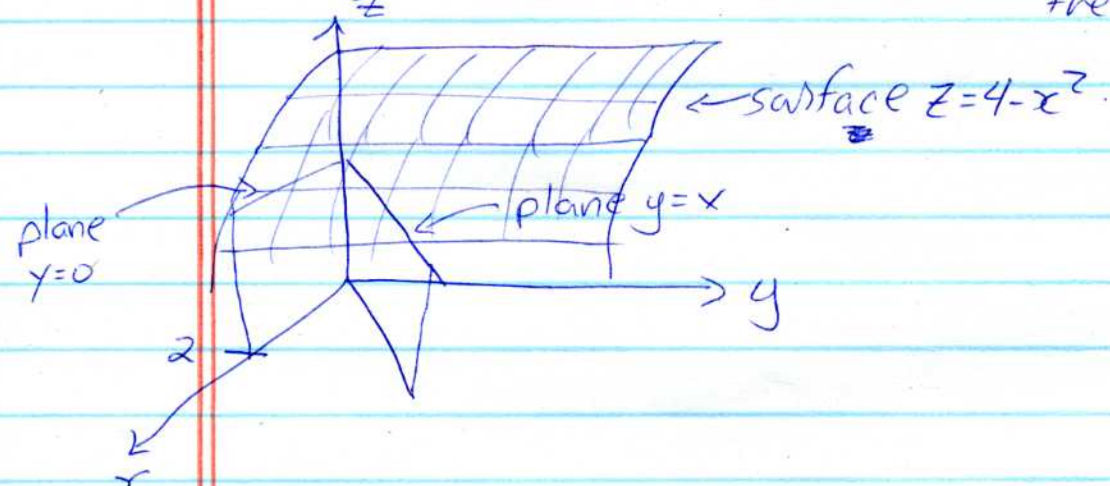
$$= 3e - 3\cancel{1} - 3 + 0$$

$$= 3e - 6$$

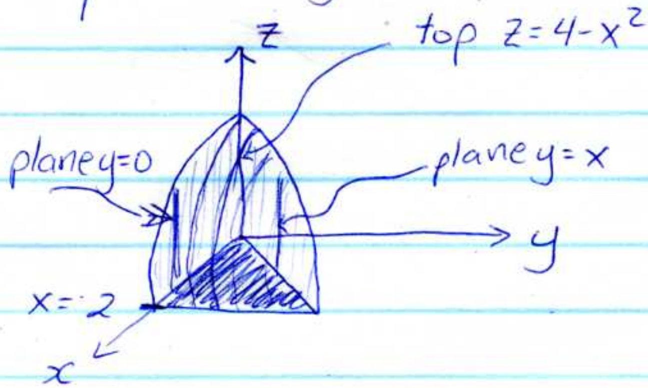
15.4.44 $\displaystyle\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\, dy\, dz\, dx$

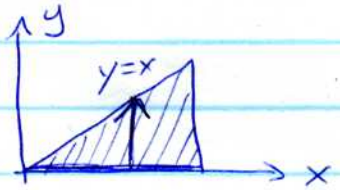
Problem: we can do the y-integral, but then we get stuck on the z-integral.

$R = \{(x,y,z) \mid 0 \le y \le x,\ 0 \le z \le 4-x^2,\ 0 \le x \le 2\}$



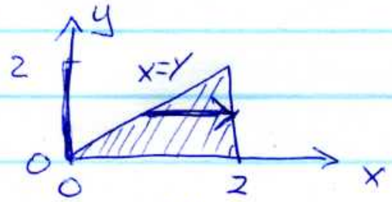

Clean up the diagram.




original description

$R = \{(x,y,z) \mid 0 \le x \le 2,\ 0 \le y \le x,\ 0 \le z \le 4-x^2\}$




ok, let's try this new description. Frankly, my hopes are not high!

$R = \{(x,y,z) \mid 0 \le y \le 2,\ y \le x \le 2,\ 0 \le z \le 4-x^2\}$

15.4.44

$$\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z}\, dy\, dz\, dx$$

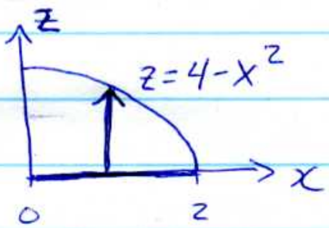
$$= \int_0^2 \int_y^2 \int_0^{4-x^2} \frac{\sin 2z}{4-z}\, dz\, dx\, dy$$

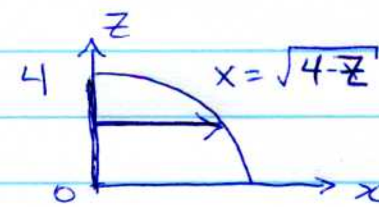
This is worse!!

Let's do the $y$-integral in the original triple integral, and see where that leads.

$$I = \int_0^2 \int_0^{4-x^2} \frac{\sin 2z}{4-z}\, x\, dz\, dx.$$

Here,




$D = \{(x,z) \mid 0 \le x \le 2,\ 0 \le z \le 4-x^2\}$ $\quad$ $D = \{(x,z) \mid 0 \le z \le 4,\ 0 \le x \le \sqrt{4-z}\}$

So $$I = \int_0^4 \int_0^{\sqrt{4-z}} \frac{\sin 2z}{4-z}\, x\, dx\, dz$$

$$= \int_0^4 \frac{\sin 2z}{4-z} \left.\frac{x^2}{2}\right|_{x=0}^{x=\sqrt{4-z}} dz$$

$$= \frac{1}{2}\int_0^4 \sin 2z\, dz$$

$$= \left.\frac{-\cos(2z)}{4}\right|_0^4 = \frac{-\cos(8)}{4} + \frac{1}{4}$$

Yeah!