16.2.14

Find the work done by $\vec{F} = \langle 2y, 3x, x+y \rangle$ over the curve $\vec{r}(t) = \langle \cos t, \sin t, t/6 \rangle$ $t=0$ to $t=2\pi$.

Use $\text{Work} = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$

$x = \cos t$
$y = \sin t$
$z = t/6$

$\vec{F} = \langle 2y, 3x, x+y \rangle$
$= \langle 2\sin t, 3\cos t, \cos t + \sin t \rangle$

$\vec{r}(t) = \langle \cos t, \sin t, t/6 \rangle$
$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1/6 \rangle$

$$\vec{F} \cdot \vec{r}\,'(t) = -2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t$$

$$\text{Work} = \int_0^{2\pi} \left(-2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$$

$$= -2\int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\,dt$$

$$+ 3\int_0^{2\pi} \frac{1}{2}(1+\cos 2t)\,dt$$

$$+ \frac{1}{6}\int_0^{2\pi} \cos t\, dt + \frac{1}{6}\int_0^{2\pi} \sin t\, dt$$

$\to 0$ $\quad \to 0$

over 1 period of $\sin t$ & $\cos t$

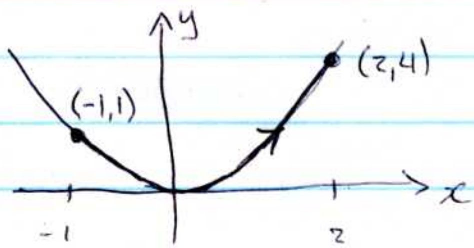
$$= \frac{1}{2}\int_0^{2\pi} dt + \frac{5}{2}\int_0^{2\pi} \cos 2t\, dt$$

$$= \pi + \frac{5}{2}\frac{\sin 2t}{2}\Big|_0^{2\pi}$$

$$= \pi.$$

16.2.17 Evaluate $\int_C xy\,dx + (x+y)\,dy$ along curve $y=x^2$ from $(-1,1)$ to $(2,4)$.

Parameterize C:



Let $x=t$ $\quad t=-1$ to $t=2$

$y=t^2$

$dx=dt$

$dy=2t\,dt$

$$\int_C xy\,dx+(x+y)\,dy = \int_{-1}^{2} (t)(t^2)(dt) + (t+t^2)(2t\,dt)$$

$$= \int_{-1}^{2} (t^3+2t^2+2t^3)\,dt$$

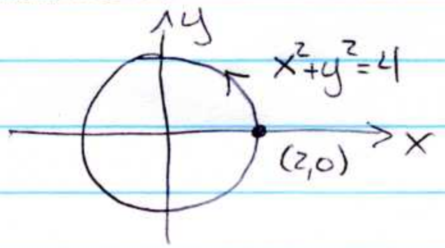
$$= \int_{-1}^{2} (3t^3+2t^2)\,dt$$

$$= \frac{3}{4}t^4 + \frac{2}{3}t^3 \Big|_{-1}^{2}$$

$$= \frac{69}{4}$$

16.2.22 Find the work done by the gradient of $f(x,y) = (x+y)^2$ counterclockwise around the circle $x^2+y^2=4$ from $(2,0)$ to itself.

Parameterize C:



$x = 2\cos t \quad 0 \le t \le 2\pi$
$y = 2\sin t$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2(x+y), 2(x+y) \rangle$$
$$= \langle 4(\cos t + \sin t), 4(\cos t + \sin t) \rangle$$

$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$
$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$

$$\nabla f \cdot \vec{r}\,'(t) = -8\cos t \sin t - 8\sin^2 t - 8\cos t \sin t - 8\cos^2 t$$
$$= -16\cos t \sin t - 8$$

$$\text{Work} = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$
$$= \int_0^{2\pi} (-16\cos t \sin t - 8)\, dt$$
$$= -8\int_0^{2\pi} dt = -8(2\pi) = -16\pi.$$

$u = \cos t$
$du = -\sin t\, dt$
when $t=0$, $u=0$
$t = 2\pi$, $u=0$
first part of integral is zero!

16.2.23 Find the circulation & flux of the fields

$$\vec{F}_1 = \langle x, y \rangle \qquad \vec{F}_2 = \langle -y, x \rangle$$

around and across each of the following curves:

a) $\vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi$

b) $\vec{r}(t) = \langle \cos t, 4\sin t \rangle \quad 0 \le t \le 2\pi$.

a) Use $\text{Flow} = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$\vec{F}_1 = \langle \cos t, \sin t \rangle$

$\vec{F}_1 \cdot \vec{r}\,'(t) = -\cos t \sin t + \cos t \sin t$
$= 0$

Flow for $\vec{F}_1$ is zero. (this is the circulation).

$\vec{F}_2 = \langle -\sin t, \cos t \rangle \qquad \vec{F}_2 \cdot \vec{r}\,'(t) = \sin^2 t + \cos^2 t = 1$

$$\text{Flow} = \int_0^{2\pi} dt = 2\pi \quad \text{for } \vec{F}_2.$$

$\vec{F}_1$ $\text{Flux} = \oint_C M\,dy - N\,dx$

$M = \cos t \quad N = \sin t$

$dy = \cos t\, dt \quad dx = -\sin t\, dt$

$$= \int_0^{2\pi} (\cos t)(\cos t\, dt) - (\sin t)(-\sin t\, dt)$$

$$= \int_0^{2\pi} dt = 2\pi.$$

$\vec{F}_2$ $\text{Flux} = \oint_C M\,dy - N\,dx$

$M = -\sin t \quad N = \cos t$

$dy = \cos t\, dt \quad dx = -\sin t\, dt$

$$= \int_0^{2\pi} (-\sin t)(\cos t\, dt) - (\cos t)(-\sin t\, dt)$$

$$= 0$$

16.2.37 $\vec{F}$ is a velocity field of a fluid. Find the flow along $\vec{r}(t) = \langle t, t^2, 1 \rangle$ $0 \le t \le 2$. $\vec{F} = \langle -4xy, 8y, z \rangle$.

Use Flow (~~or circulation~~) $= \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$

$\vec{r}(t) = \langle t, t^2, 1 \rangle \Rightarrow x = t,\ y = t^2,\ z = 1$

$\vec{r}\,'(t) = \langle 1, 2t, 0 \rangle$

$\vec{F} = \langle -4xy, 8y, z \rangle$

$= \langle -4t^3, 8t^2, 2 \rangle$

$\vec{F} \cdot \vec{r}\,'(t) = -4t^3 + 16t^3 = 12t^3.$

Flow ~~Circulation~~ $= \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$

$= \int_0^2 12t^3\, dt$

$= \frac{12t^4}{4}\Big|_0^2 = 48.$