16.3.11

$$\vec{F} = \left\langle \ln x + \sec^2(x+y),\ \sec^2(x+y) + \frac{y}{y^2+z^2},\ \frac{z}{y^2+z^2} \right\rangle$$

$$= \nabla f$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\Rightarrow \quad f = \int \ln x + \sec^2(x+y)\, dx$$

$$= x\ln x - x + \tan(x+y) + g(y,z)$$

Basic form $\int \sec^2\theta = \tan\theta + C$

$\int \ln x\, dx$ using parts

$$f = \int \sec^2(x+y) + \frac{y}{y^2+z^2}\, dy$$

usub $u = y^2+z^2$ in second integral

$$= \tan(x+y) + \frac{1}{2}\ln(y^2+z^2) + h(x,z)$$

$$f = \int \frac{z}{y^2+z^2}\, dz$$

usub $u = y^2+z^2$.

$$= \frac{1}{2}\ln(y^2+z^2) + \omega(x,y)$$

Comparing our expressions for f, we see

$$f(x,y,z) = x\ln x - x + \tan(x+y) + \frac{1}{2}\ln(y^2+z^2) + C$$

16.3.20 Find a potential function and evaluate

$$\int_{(2,1,1)}^{(1,2,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$$

$y > 0.$

$$\vec{F} = \left\langle 2x\ln y - yz,\ \frac{x^2}{y} - xz,\ -xy \right\rangle$$

Check $\vec{F}$ is conservative using component test:

$$\frac{\partial}{\partial y}(-xy) = -x \quad \frac{\partial}{\partial z}\left(\frac{x^2}{y} - xz\right) = -x \quad ✓$$

$$\frac{\partial}{\partial z}(2x\ln y - yz) = -y \quad \frac{\partial}{\partial x}(-xy) = -y \quad ✓$$

$$\frac{\partial}{\partial x}\left(\frac{x^2}{y} - xz\right) = 2x/y \quad \frac{\partial}{\partial y}(2x\ln y - yz) = 2x/y \quad ✓$$

Conservative ⇒ exact!

$$\frac{\partial f}{\partial x} = 2x\ln y - yz \Rightarrow f = x^2\ln y - xyz + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} - xz \Rightarrow f = x^2\ln y - xyz + h(x,z)$$

$$\frac{\partial f}{\partial z} = -xy \Rightarrow f = -xyz + \omega(x,y)$$

comparing, we see $f(x,y,z) = x^2\ln y - xyz + C$

∴ The integral evaluates to

$$f(1,2,1) - f(2,1,1) = -2 + \ln 2 + 2 = \ln 2.$$

16.3.23 $\displaystyle\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$ along line from $(1,1,1)$ to $(2,3,-1)$.

$$= \int_{(1,1,1)}^{(2,3,-1)} \vec{F}\cdot d\vec{r} = \int_{a}^{b} \vec{F}\cdot\vec{r}\,'\,dt$$

where $\vec{F} = \langle y, x, 4\rangle$ $\qquad \vec{r}(t) = \langle 1,1,1\rangle + t\langle 1,2,-2\rangle \quad 0 \le t \le 1$

$= \langle 1+2t, 1+t, 4\rangle \qquad = \langle 1+t, 1+2t, 1-2t\rangle$

$\vec{r}\,'(t) = \langle 1, 2, -2\rangle$

$\vec{F}\cdot\vec{r}\,' = 1+2t + 2(1+t) + 4(-2)$

$= -5 + 4t$

$$\text{integral} = \int_0^1 (-5+4t)\,dt$$

$$= \left(-5t+2t^2\right)\Big|_0^1$$

$$= -3$$

16.3.26) $\displaystyle\int_A^B \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$ $\qquad \vec{F} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$

Independence of Path $\longleftrightarrow$ conservative $\longleftrightarrow$ exact.

Show $\dfrac{\partial P}{\partial y} = \dfrac{\partial N}{\partial z}$, $\quad \dfrac{\partial M}{\partial z} = \dfrac{\partial P}{\partial x}$, $\quad \dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$

$$\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-yz}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) \quad \checkmark$$

$$\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-xz}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \quad \checkmark$$

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-xy}{\sqrt{x^2+y^2+z^2}} = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) \quad \checkmark$$

It's conservative! So the integral doesn't depend on the path from A to B.