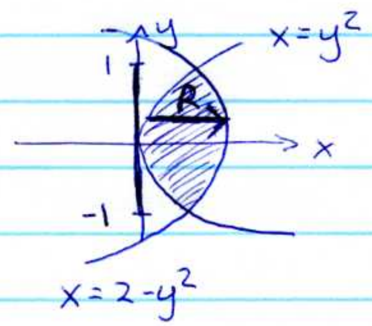
16.5.3 Find the area of the region cut from the plane $x+2y+2z=5$ by cylinders whose walls are $x=y^2$ and $x=2-y^2$.

Region:



Intersection

$$x = y^2 = 2 - y^2$$

$$y = \pm 1$$

$$R = \{ (x,y) \mid -1 \le y \le 1,\ y^2 \le x \le 2-y^2 \}$$

$$x+2y+2z=5 \Rightarrow z = \frac{5-x-2y}{2}$$

$$\vec{r}(x,y) = \left\langle x, y, \frac{5-x-2y}{2} \right\rangle$$

Note: if you prefer, parameterize using $x=u,\ y=v$.

$$\vec{r}_x = \langle 1, 0, -1/2 \rangle$$

$$\vec{r}_y = \langle 0, 1, -1 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1/2 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1/2, 1, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\tfrac{1}{4}+1+1} = \frac{3}{2}$$

$$\text{Surface area} = \iint_R d\sigma$$

$$= \iint_R |\vec{r}_x \times \vec{r}_y| \, dx\,dy$$

$$= \frac{3}{2}\int_{-1}^{1}\int_{y^2}^{2-y^2} dx\,dy$$

$$= \frac{3}{2}\int_{-1}^{1} (2-y^2-y^2)\,dy$$

$$= \frac{3}{2}\int_{-1}^{1} (2-2y^2)\,dy$$

$$= 3\left(y - \frac{y^3}{3}\right)\Big|_{-1}^{1}$$

$$= 3\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$$= 3\left(\frac{4}{3}\right) = 4.$$

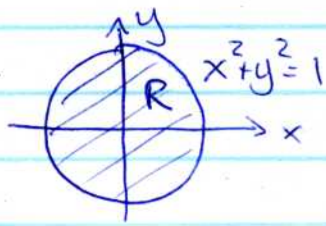
16.5.6 Find the area of the cap cut from the sphere $x^2+y^2+z^2=2$ by the cone $z=\sqrt{x^2+y^2}$

We need intersection of these two surfaces:

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=2$$

$$x^2+y^2=1 \Rightarrow \text{circle in } xy \text{ plane!}$$

(this is a shadow region).



$x=r\cos\theta$
$y=r\sin\theta$
$0\le r\le 1$
$0\le\theta<2\pi$.

surface: $z=\sqrt{2-x^2-y^2}$
$=\sqrt{2-r^2}$

Parameterize surface: $\vec{r}(r,\theta)=\langle x,y,z\rangle$
$=\langle r\cos\theta,\ r\sin\theta,\ \sqrt{2-r^2}\rangle$

$$\vec{r}_r=\left\langle \cos\theta,\ \sin\theta,\ \frac{-r}{\sqrt{2-r^2}}\right\rangle$$

$$\vec{r}_\theta=\langle -r\sin\theta,\ r\cos\theta,\ 0\rangle$$

$$\vec{r}_r\times\vec{r}_\theta=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{2-r^2}}\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}=\left\langle \frac{-r^2\cos\theta}{\sqrt{2-r^2}},\ \frac{-r^2\sin\theta}{\sqrt{2-r^2}},\ r\right\rangle$$

$$|\vec{r}_r\times\vec{r}_\theta|=\sqrt{\frac{r^4}{2-r^2}+r^2}=\sqrt{\frac{r^4+2r^2-r^4}{2-r^2}}=\frac{\sqrt{2}\,r}{\sqrt{2-r^2}}$$

Surface area $=\iint_R d\sigma$

$=\iint_R |\vec{r}_r\times\vec{r}_\theta|\,dr\,d\theta$

$=\int_0^{2\pi}\int_0^1 \frac{\sqrt{2}\,r}{\sqrt{2-r^2}}\,dr\,d\theta$

$u=2-r^2$
$du=-2r\,dr$
when $r=0$ $u=2$
$r=1$ $u=1$

$=2\pi\frac{\sqrt{2}}{(-2)}\int_2^1\frac{du}{u^{1/2}}$

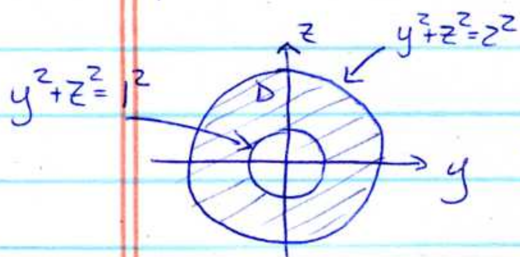
$=\frac{2\pi\sqrt{2}}{-2}\left.\frac{u^{1/2}}{(1/2)}\right|_2^1$

$=-2\sqrt{2}\pi\left(1-\sqrt{2}\right)$.

16.5.9

Find the area of the portion of the paraboloid $x = 4 - y^2 - z^2$ that lies above the ring $1 \le y^2 + z^2 \le 4$ in the yz-plane.

parameterize $1 \le y^2 + z^2 \le 4$ and use surface area $= \iint_D |\vec{r}_u \times \vec{r}_v| \, du\, dv$



$y = r\cos\theta$

$z = r\sin\theta$

$1 \le r \le 2$

$0 \le \theta < 2\pi$

so $D = \{ (y,z) = (r\cos\theta, r\sin\theta) \mid 1 \le r \le 2,\ 0 \le \theta < 2\pi \}$

$x = 4 - y^2 - z^2 = 4 - r^2$

note: this notation is ok if you are careful — $\vec{r}$ is different from r!

$\vec{r}(r,\theta) = \langle x, y, z \rangle = \langle 4 - r^2, r\cos\theta, r\sin\theta \rangle$

$\vec{r}_r = \langle -2r, \cos\theta, \sin\theta \rangle$

$\vec{r}_\theta = \langle 0, -r\sin\theta, r\cos\theta \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2r & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{vmatrix} = \langle r, +2r^2\cos\theta, 2r^2\sin\theta \rangle$$

$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2 + 4r^4} = r\sqrt{1+4r^2}$

Surface Area $= \iint_D |\vec{r}_r \times \vec{r}_\theta| \, dr\, d\theta$

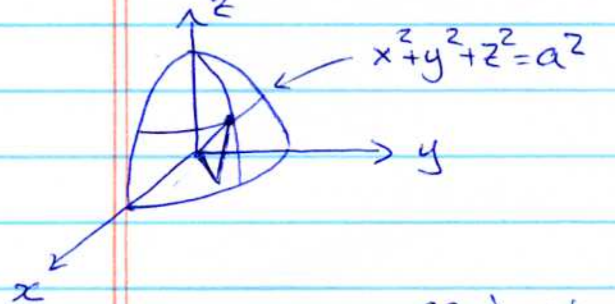
$= \int_0^{2\pi}\int_1^2 r\sqrt{1+4r^2}\, dr\, d\theta$ let $u = 1+4r^2$, $du = 8r\,dr$, when $r=1, u=5$; $r=2, u=17$

$= 2\pi \frac{1}{8} \int_5^{17} u^{1/2} du$

$= \frac{\pi}{4} \left. \frac{u^{3/2}}{(3/2)} \right|_5^{17}$

$= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$

16.5.25 Find the flux of the field $\vec{F} = \langle x, y, z \rangle$ across the portion of the sphere $x^2+y^2+z^2=a^2$ in 1st octant away from origin.



Use $\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\,d\sigma$

$d\sigma = |\vec{r}_u \times \vec{r}_v|\,du\,dv$

$\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

(since $\vec{r}_u \times \vec{r}_v$ is normal to tangent plane).

$$\text{Flux} = \iint_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv$$

Parameterize surface (sphere):

$\vec{r} = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$ $\quad 0 \le \phi \le \pi/2$, $0 \le \theta \le \pi/2$

$\vec{r}_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi \rangle$

$\vec{r}_\theta = \langle -a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0 \rangle$

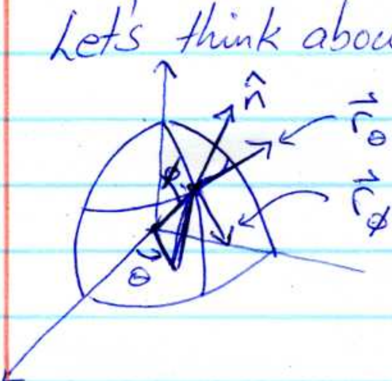
$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \langle a^2\sin^2\phi\cos\theta,\ +a^2\sin^2\phi\sin\theta,\ a^2\cos\phi\sin\phi \rangle$$

$|\vec{r}_\phi \times \vec{r}_\theta| =$ hey! We don't need this!

16.5.25 continued.

Let's think about orientation $\hat{n}$, which should point outwards.



So outward normal is

$$\hat{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{|\vec{r}_\phi \times \vec{r}_\theta|} \quad \text{(use right hand rule)}$$

OK, so direction is correct.

use $\vec{r}$ to get $x = a\sin\phi\cos\theta$ etc in $\vec{F}$.

Simplify integrand:

$$\vec{F}\cdot(\vec{r}_\phi \times \vec{r}_\theta) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi\rangle \cdot \langle a^2\sin^2\phi\cos\theta,\ +a^2\sin^2\phi\sin\theta,\ a^2\cos\phi\sin\phi\rangle$$

$$= a^3\sin^3\phi\cos^2\theta + a^3\sin^3\phi\sin^2\theta + a^3\cos^2\phi\sin\phi$$

$$= a^3\sin^3\phi(\cos^2\theta + \sin^2\theta) + a^3\cos^2\phi\sin\phi$$

$$= a^3\sin^3\phi + a^3\cos^2\phi\sin\phi.$$

Now, we can do the flux:

$$\text{Flux} = \iint_S \vec{F}\cdot(\vec{r}_\phi \times \vec{r}_\theta)\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} a^3\sin^3\phi + a^3\cos^2\phi\sin\phi\ d\phi\,d\theta$$

$$= a^3\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{\pi/2}d\theta + a^3\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta$$

Do each integral in turn.

$$\int_0^{\pi/2}\sin^3\phi\,d\phi = \int_0^{\pi/2}\sin^2\phi\sin\phi\,d\phi$$

$$= \int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi$$

$$= \int_0^{\pi/2}\sin\phi\,d\phi - \underbrace{\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi}_{\substack{u=\cos\phi \\ du=-\sin\phi\,d\phi}}$$

when $\phi=0$ $u=1$, $\phi=\pi/2$ $u=0$

$$= -\cos\phi\Big|_0^{\pi/2} + \int_1^0 u^2\,du$$

$$= 1 + \frac{u^3}{3}\Big|_1^0 = 1 - \frac{1}{3} = \frac{2}{3}$$

16.5.25
continued

As before, $\int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi = \frac{1}{3}$

and $\int_0^{\pi/2} d\theta = \pi/2$.

Put it all back together:

$$\text{Flux} = a^3\left(\frac{2}{3}\right)\left(\pi/2\right) + a^3\left(\frac{1}{3}\right)\left(\pi/2\right)$$

$$= \frac{a^3\pi}{2}.$$