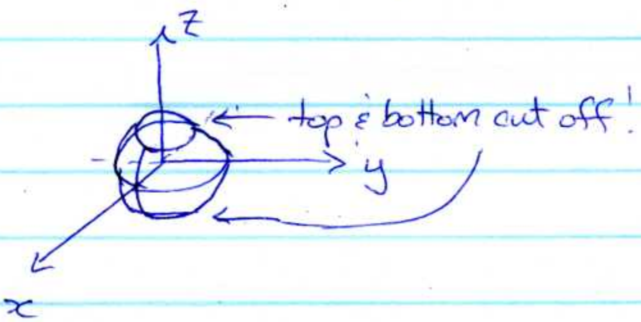
$\boxed{16.6.7}$ Parameterize the portion of the sphere $x^2+y^2+z^2=3$ between the planes $z=\frac{\sqrt{3}}{2}$ and $z=\frac{-\sqrt{3}}{2}$.

Trying spherical coordinates makes sense.

$$x=\sqrt{3}\sin\phi\cos\theta \qquad 0\le\theta\le 2\pi$$
$$y=\sqrt{3}\sin\phi\sin\theta$$
$$z=\sqrt{3}\cos\phi.$$

Need bounds on ϕ.



top: $z=\frac{\sqrt{3}}{2}=\sqrt{3}\cos\phi$

$\frac{1}{2}=\cos\phi$

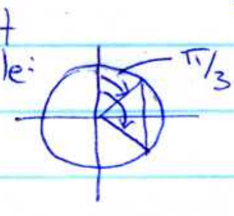
$\cos\pi/3=\frac{1}{2}$

$\Rightarrow \phi=\pi/3.$

bottom: $z=-\frac{\sqrt{3}}{2}=\sqrt{3}\cos\phi$

$-\frac{1}{2}=\cos\phi$

unit circle:



$\hookrightarrow \phi=\pi-\pi/3$
$=2\pi/3.$

$\Rightarrow \pi/3\le\phi\le\frac{2\pi}{3}.$

Parameterization of surface

$$\vec{r}(\phi,\theta)=\langle\sqrt{3}\sin\phi\cos\theta,\ \sqrt{3}\sin\phi\sin\theta,\ \sqrt{3}\cos\phi\rangle$$
$$0\le\theta<2\pi$$
$$\pi/3\le\phi\le 2\pi/3.$$

16.6.13 Parameterize the postion of the plane $x+y+z=1$

a) inside cylinder $x^2+y^2=9$

b) inside cylinder $y^2+z^2=9$.

a) Cylinders $\Rightarrow$ use polar coordinates on xy-plane.

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ 0 \le \theta < 2\pi \\ 0 \le r \le 3 \end{array}\right\}$$

Now solve for surface $z = 1-x-y$
$= 1 - r\cos\theta - r\sin\theta$

So surface is $\vec{r}(r,\theta) = \langle x, y, z \rangle$
$= \langle r\cos\theta, r\sin\theta, 1 - r\cos\theta - r\sin\theta \rangle$
$0 \le \theta < 2\pi, \ 0 \le r \le 3$.

b) use polar coordinates in yz-plane

$$\left.\begin{array}{l} y = r\cos\theta \\ z = r\sin\theta \\ 0 \le \theta < 2\pi \\ 0 \le r \le 3 \end{array}\right\}$$

Now solve for surface $x = 1-y-z$
$= 1 - r\cos\theta - r\sin\theta$

So surface is $\vec{r}(r,\theta) = \langle x, y, z \rangle$
$= \langle 1 - r\cos\theta - r\sin\theta, r\cos\theta, r\sin\theta \rangle$
$0 \le \theta < 2\pi, \ 0 \le r \le 3$.

16.6.15 Parameterize the portion of cylinder $(x-2)^2 + z^2 = 4$ between $y=0$ & $y=3$.

Modify polar: $x = r\cos\theta + 2$
$z = r\sin\theta$

and now I'm stuck, because I've used two parameters - need to stop and think.

The surface is on the cylinder, so I do not want the parameter r in above - that will give the inside of cylinder which I don't need or want.

OK:
$$\left.\begin{array}{l} x = 2\cos\theta + 2 \\ z = 2\sin\theta \end{array}\right\} \text{satisfies } (x-2)^2 + z^2 = 4, \quad 0 \le \theta < 2\pi$$
$$y = u \qquad 0 \le u \le 3$$

Surface: $\vec{r}(\theta, u) = \langle x, y, z \rangle$
$$= \langle 2\cos\theta + 2,\ u,\ 2\sin\theta \rangle$$
$$0 \le \theta < 2\pi,\ 0 \le u \le 3.$$

16.6.17 Find surface area of the portion of the plane $y + 2z = 2$ inside $x^2+y^2=1$.

Parameterize cylinder $x^2+y^2 \le 1$ $\quad x = r\cos\theta \quad 0 \le r \le 1$

$y = r\sin\theta \quad 0 \le \theta < 2\pi$

surface $z = 1 - y/2$

surface $\vec{r}(r,\theta) = \langle x,y,z \rangle = \langle r\cos\theta, r\sin\theta, 1 - \frac{r}{2}\sin\theta \rangle$

$$\text{surface area} = \iint_S d\sigma = \int_0^{2\pi}\int_0^1 |\vec{r}_r \times \vec{r}_\theta| \, dr\, d\theta$$

$\vec{r}_r = \langle \cos\theta, \sin\theta, -\frac{1}{2}\sin\theta \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, -\frac{r}{2}\cos\theta \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -\frac{1}{2}\sin\theta \\ -r\sin\theta & r\cos\theta & -\frac{r}{2}\cos\theta \end{vmatrix} = \langle 0, -r, -\frac{r}{2} \rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2 + \frac{r^2}{4}} = \frac{\sqrt{5}\,r}{2}$$

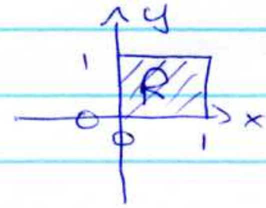
$$\text{surface area} = \int_0^{2\pi}\int_0^1 \frac{\sqrt{5}\,r}{2}\, dr\, d\theta$$

$$= 2\pi \left.\frac{\sqrt{5}\,r^2}{4}\right|_0^1$$

$$= \frac{\pi\sqrt{5}}{2}.$$

16.6.35 Integrate $G(x,y,z) = z$ over the portion of the plane $x+y+z=4$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$ in xy-plane.

Region in xy-plane:



shadow region, so we can use x, y as the parameters!

plane is $z = 4 - x - y$.

Surface is $\vec{r}(x,y) = \langle x, y, 4-x-y \rangle$ $\quad 0 \le x \le 1$, $0 \le y \le 1$

Note: if you don't like using x, y as parameters, switch to $\vec{r}(u,v) = \langle u, v, 4-u-v \rangle$. $0 \le u \le 1$, $0 \le v \le 1$.

We need $\iint_S G(x,y,z)\, d\sigma = \int_0^1\int_0^1 z\, |\vec{r}_x \times \vec{r}_y|\, dx\, dy$

$\vec{r}_x = \langle 1, 0, -1 \rangle$

$\vec{r}_y = \langle 0, 1, -1 \rangle$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{3}$

$= \int_0^1\int_0^1 (4-x-y)\sqrt{3}\, dx\, dy$

$= 3\sqrt{3}$

16.6.41 Find the Flux of $\vec{F} = \langle xy, 0, -z \rangle$ outward through the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$.

$\text{Flux} = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv \Longrightarrow$ (need to check we get outward normal!)

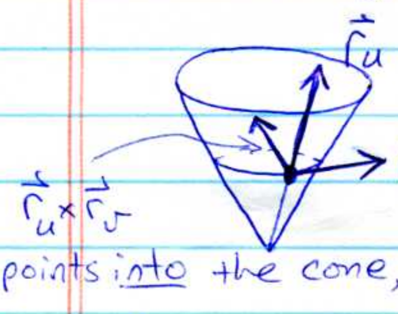
Parameterize cone $\vec{r}(u,v) = \langle u\cos v, u\sin v, u \rangle$ $\quad 0 \le u \le 1$, $0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -u\cos v, -u\sin v, u \rangle$

Check if this gives outward pointing normal:



points into the cone, not out!

We should use $\vec{r}_v \times \vec{r}_u$ to get outward pointing normal.
$\vec{r}_v \times \vec{r}_u = \langle u\cos v, u\sin v, -u \rangle$.

$\vec{F} \cdot (\vec{r}_v \times \vec{r}_u) = \langle u^2\cos v \sin v, 0, -u \rangle \cdot \langle u\cos v, u\sin v, -u \rangle$
$= u^3\cos^2 v \sin v + u^2$

$\text{Flux} = \int_0^{2\pi}\int_0^1 (u^3\cos^2 v \sin v + u^2)\, du\, dv$

$= \underbrace{\frac{1}{4}\int_0^{2\pi} \cos^2 v \sin v\, dv}_{\text{let } \omega = \cos v \text{ and change to } \omega\text{-integration. This integral is zero!}} + 2\pi/3$

$= 2\pi/3$.