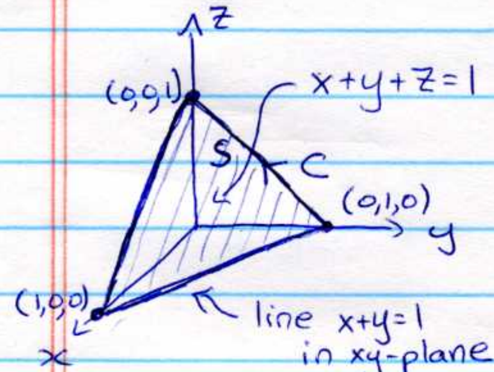
16.7.3 | Calculate the circulation of the field $\vec{F} = \langle y, xz, x^2 \rangle$ around curve C: triangle cut from plane $x+y+z=1$ by 1st octant, counterclockwise when viewed from above.



$S: \vec{r}(u,v) = \langle u, v, 1-u-v \rangle$

$0 \le u \le 1, \quad 0 \le v \le 1-u$

$\vec{F} = \langle y, xz, x^2 \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix}$$

$= \langle -x, -2x, z-1 \rangle$

$= \langle -u, -2u, 1-u-v-1 \rangle$

$\nabla \times \vec{F} = \langle -u, -2u, -u-v \rangle$

$\vec{r}_u = \langle 1, 0, -1 \rangle$

$\vec{r}_v = \langle 0, 1, -1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= \langle 1, 1, 1 \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{3}$

$\hat{n} = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle \left(= \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right)$

oriented upwards, which is what we want for counterclockwise traversal of C.

$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du\,dv = \sqrt{3}\, du\,dv$

Circulation $= \oint_C \vec{F} \cdot d\vec{r}$

$= \iint_S \nabla \times \vec{F} \cdot \hat{n} \, d\sigma$

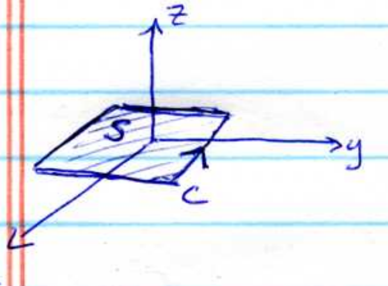
$= \int_0^1 \int_0^{1-u} \langle -u, -2u, -u-v \rangle \cdot \langle 1,1,1 \rangle \, dv\,du$

$= \int_0^1 \int_0^{1-u} (-u-2u-u-v) \, dv\,du$

$= \int_0^1 \left(-\frac{1}{2} - 3u + \frac{7u^2}{2} \right) du$

$= -\frac{5}{6}$

16.7.5 | Calculate the circulation of the field $\vec{F} = \langle y^2+z^2, x^2+y^2, x^2+y^2 \rangle$ around the curve C: square bounded by lines $x = \pm 1$, $y = \pm 1$, counterclockwise when viewed from above.



$$\text{Circulation} = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \iint_S \nabla \times \vec{F} \cdot \hat{n}\, d\sigma$$

$$= \int_{-1}^{1}\int_{-1}^{1} \langle 2v, -2u, 2u \mp 2v \rangle \cdot \langle 0,0,1 \rangle\, du\, dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} (2u \mp 2v)\, du\, dv$$

$$= \int_{-1}^{1} dv \int_{-1}^{1} 2u\, du \mp \int_{-1}^{1} du \int_{-1}^{1} 2v\, dv$$

$$= 2\, u^2 \Big|_{-1}^{1} \mp 2\, v^2 \Big|_{-1}^{1}$$

$$= 0.$$

initial computations!

$S: \vec{r}(u,v) = \langle u, v, 0 \rangle$
$-1 \le u \le 1$
$-1 \le v \le 1$

$\vec{F} = \langle y^2+z^2, x^2+y^2, x^2+y^2 \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix}$$

$= \langle 2y, -2x+2z, 2x \mp 2y \rangle$
$= \langle 2v, -2u, 2u \mp 2v \rangle$

$\vec{r}_u = \langle 1,0,0 \rangle$
$\vec{r}_v = \langle 0,1,0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 0,0,1 \rangle$$

$|\vec{r}_u \times \vec{r}_v| = 1$

$\hat{n} = \langle 0,0,1 \rangle$ (oriented upwards)
$\Rightarrow$ counterclockwise path around C.

$d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv = du\, dv$