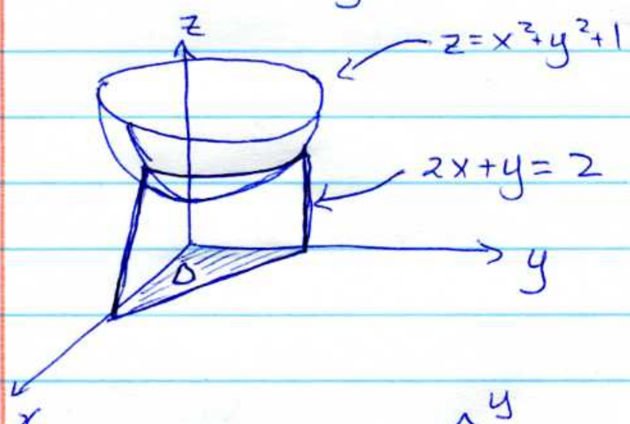
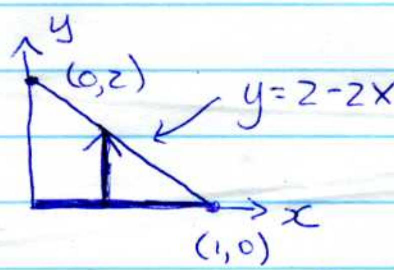
Ex Find the volume V of the solid in the first octant bounded by the coordinate planes and the graphs of $z = x^2+y^2+1$ and $2x+y=2$.

Solution 1 $z = x^2+y^2+1$ is a paraboloid.
$2x+y=2$ is a plane.



In xy-plane :



$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2-2x\}$

Also $z = f(x,y) = x^2+y^2+1$.

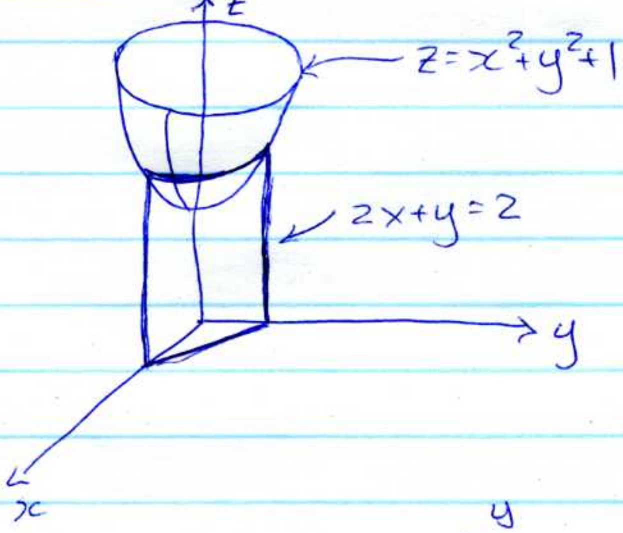
$$V = \iint_D f(x,y)\,dA$$

$$= \int_0^1 \int_0^{2-2x} (x^2+y^2+1)\,dy\,dx$$

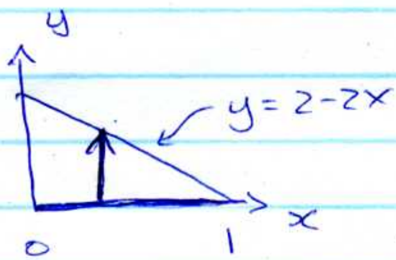
$$= \int_0^1 \left(\frac{14}{3} - 10x + 10x^2 - \frac{14x^3}{3} \right) dx$$

$$= \frac{11}{6}$$

Solution 2 | We still need the sketch.



In xy-plane



Region E is the volume:

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 2-2x,\ 0 \le z \le x^2+y^2+1\}$$

$$V = \iiint_E dV$$

$$= \int_0^1 \int_0^{2-2x} \int_0^{x^2+y^2+1} dz\,dy\,dx$$

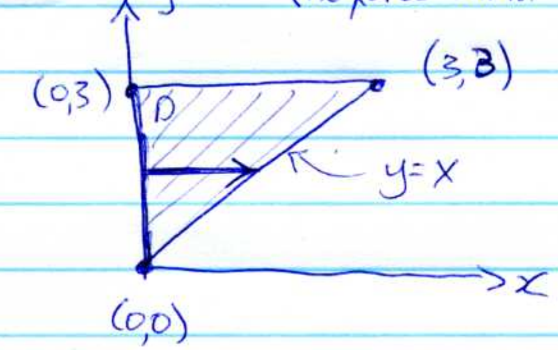
$$= \int_0^1 \int_0^{2-2x} (x^2+y^2+1)\,dy\,dx \quad \text{same as before}$$

$$= \frac{11}{6}$$

Ex) $I = \iint_D \sqrt{x^2+y^2}\, dA$ where D is triangular region with vertices $(0,0)$, $(0,3)$, and $(3,3)$.

(Requies MMA to do integrals)

Solution)



$D = \{(x,y) \mid 0 \le y \le 3,\ 0 \le x \le y\}$

$$I = \int_0^3 \int_0^y \sqrt{x^2+y^2}\, dx\, dy$$

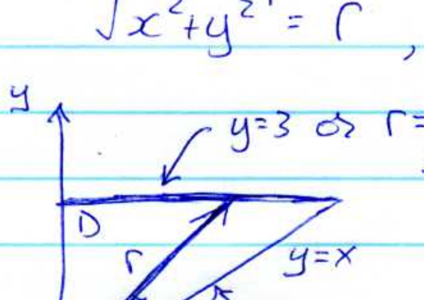
MMA.

$$= \frac{9}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)$$

Note: this integral is difficult to do by hand. It would require using a substitution of
$x = y \sinh\theta$
(hyperbolic sine).

Let's try switching to polar: use $dA = r\,dr\,d\theta$

$\sqrt{x^2+y^2} = r$,



$D = \{(r,\theta) \mid \pi/4 \le \theta \le \pi/2,\ 0 \le r \le 3\csc\theta\}$

$$I = \int_{\pi/4}^{\pi/2} \int_0^{3\csc\theta} r^2\, r\, d\theta$$

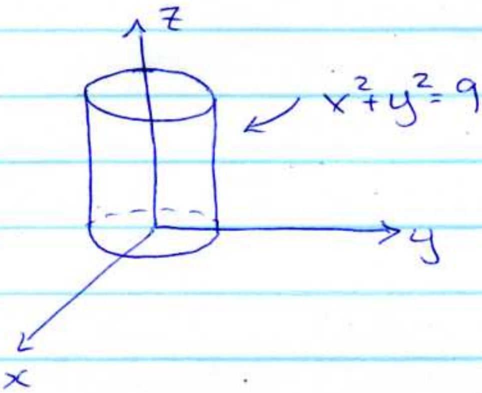
$$= \int_{\pi/4}^{\pi/2} 9\csc^3\theta\, d\theta$$

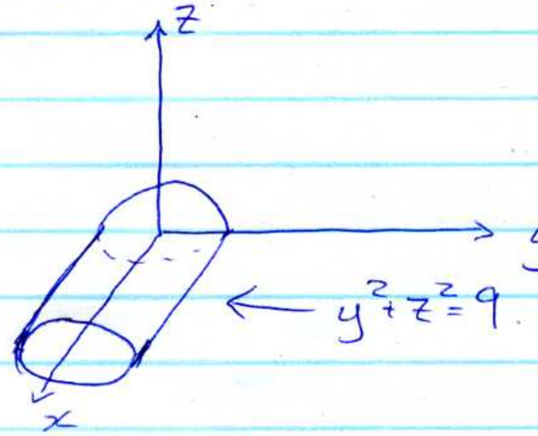
$$= \frac{9}{2}\left(\sqrt{2} - \ln\left(\tan(\pi/8)\right)\right)$$

Again, a pretty involved integration by hand

Ex Find the volume V of the solid bounded by $x^2+y^2=9$ and $y^2+z^2=9$.

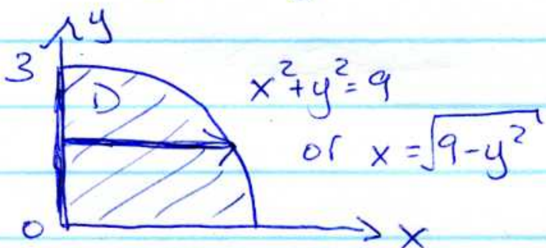
Solution Both surfaces are cylinders of radius 3.





Use symmetry: work in first octant & multiply by 8.

Volume lies over $x^2+y^2=9$, and function is
$z=f(x,y)=\sqrt{9-y^2}$



$D=\{(x,y)\mid 0\le x\le\sqrt{9-y^2},\ 0\le y\le 3\}$

$$V=8\int_0^3\int_0^{\sqrt{9-y^2}}\sqrt{9-y^2}\,dx\,dy$$

$$=8\int_0^3 \sqrt{9-y^2}\,x\Big|_0^{\sqrt{9-y^2}}dy$$

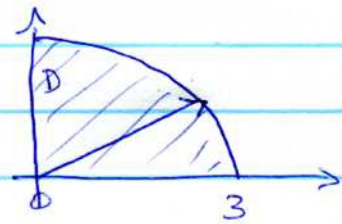
$$=8\int_0^3 (9-y^2)\,dy$$

$$=8\left(9y-\frac{y^3}{3}\right)\Big|_0^3$$

$$=8\left(27-\frac{27}{3}\right)$$

$$=144.$$

Note: if you switch to polar, the region is easier to describe but the integral is harder to do!



$$D = \{(r,\theta) \mid 0 \le r \le 3,\ 0 \le \theta \le \pi/2\}$$

$$f(x,y) = f(r\cos\theta, r\sin\theta) = \sqrt{9 - r^2\sin^2\theta}$$

$$V = 8\int_0^{\pi/2}\int_0^3 \sqrt{9 - r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$= 8\int_0^{\pi/2}\int_9^{9\cos^2\theta} \sqrt{u}\,\frac{du\,d\theta}{(-2\sin^2\theta)}$$

$$= 8\int_0^{\pi/2} \frac{1}{2}\,\frac{1}{\sqrt{u}\,(-2\sin^2\theta)}\Bigg|_{u=9}^{u=9\cos^2\theta} d\theta$$

$$= \frac{8}{-4}\int_0^{\pi/2}\left(\frac{1}{3\cos\theta\sin^2\theta} - \frac{1}{3\sin^2\theta}\right)d\theta$$

substitution:

let $u = 9 - r^2\sin^2\theta$

$du = -2r\sin^2\theta\, dr$

when $r=0$, $u = 9$

$r = 3$ $u = 9 - 9\sin^2\theta$

$= 9\cos^2\theta$

= yuck!! We've already got the answer. Let's stop this madness!!