15.4 Triple Integrals

$$\iiint_B f(x,y,z)\,dV = \lim_{l,n,m\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

where $\Delta V = \Delta x \Delta y \Delta z$

If $B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Fubini's Theorem states that we can reorder this

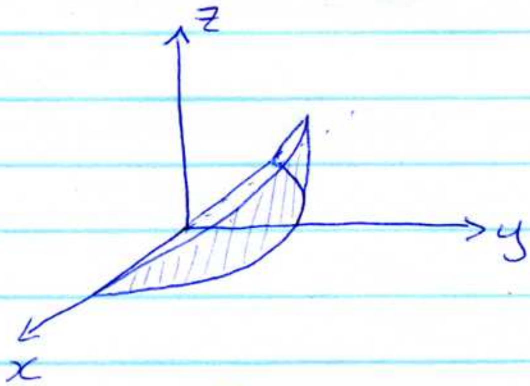
$$\left.\begin{matrix} dx\,dz\,dy \\ dz\,dx\,dy \\ \text{etc} \end{matrix}\right\} \text{6 equivalent triple integrals.}$$

Note: $dV = dx\,dy\,dz$.

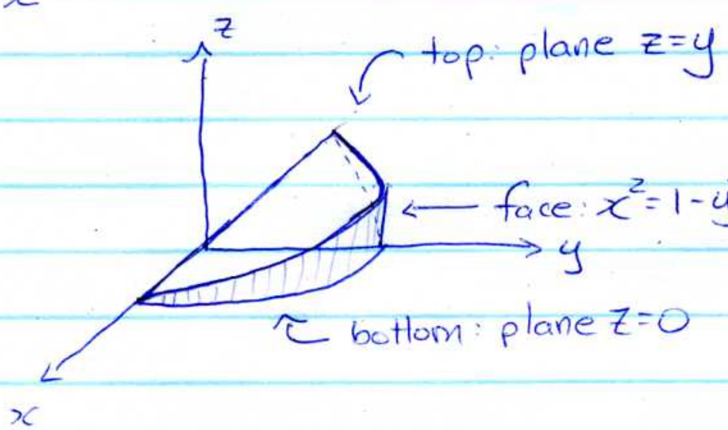
$\iiint_B dV$ is the volume of the region B in $\mathbb{R}^3$.

Ex Express $\iiint_E dV$ as 6 different iterated integrals, where E is solid bounded by $z=0$, $z=y$, $x^2=1-y$

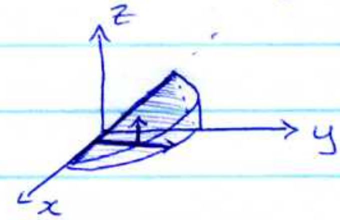
Sketch region:



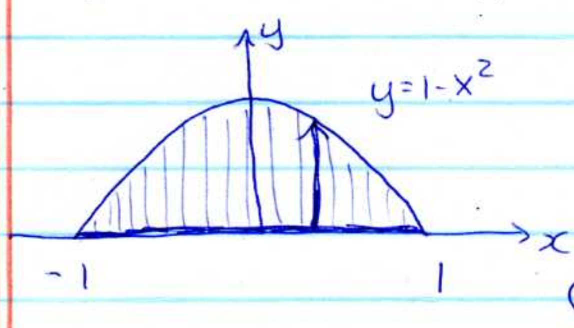
Redraw:



We get different integral representations by projecting onto different planes.

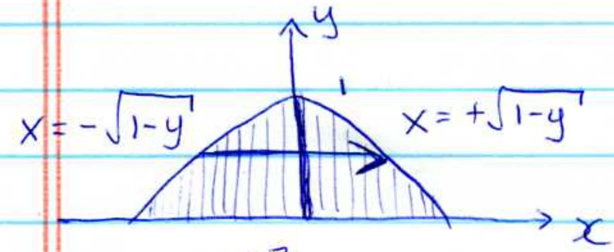


Project onto xy-plane

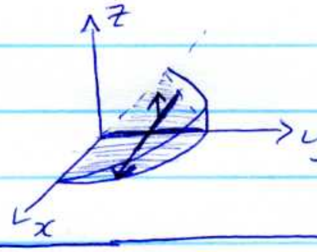


$E = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le y \le 1-x^2,\ 0 \le z \le y\}$

$$\iiint_E dV = \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} dz\,dy\,dx$$

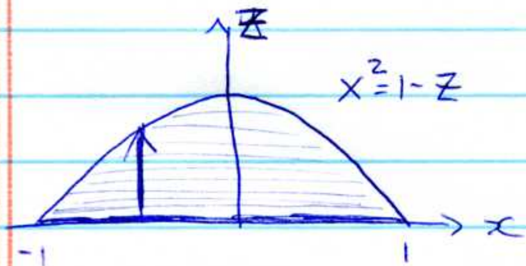


$$E = \{(x,y,z) \mid 0 \le y \le 1,\ -\sqrt{1-y} \le x \le +\sqrt{1-y},\ 0 \le z \le y\}$$



$$\iiint_E dV = \int_0^1 \int_{-\sqrt{1-y}}^{+\sqrt{1-y}} \int_0^y dz\,dx\,dy$$

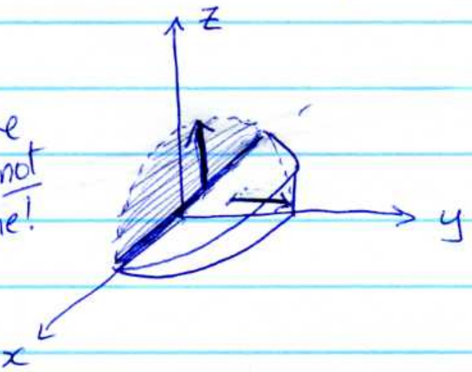
$\boxed{\text{Project onto } xz\text{-plane}}$

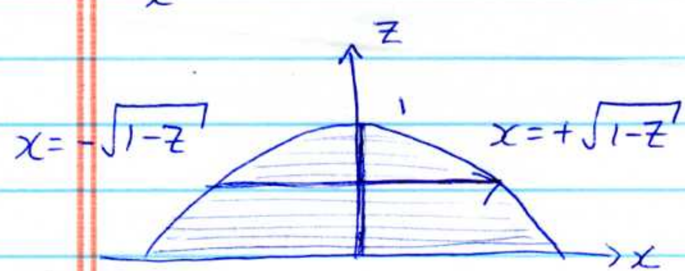


$$E = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le z \le 1-x^2,\ z \le y \le 1-x^2\}$$

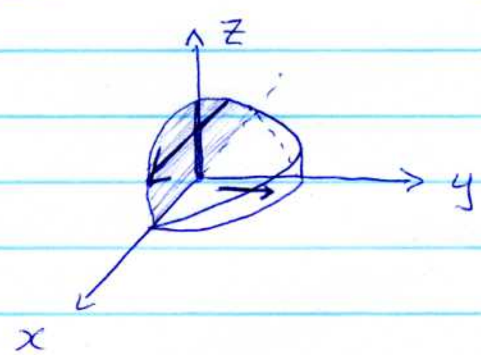
$$\iiint_E dV = \int_{-1}^{1} \int_0^{1-x^2} \int_z^{1-x^2} dy\,dz\,dx$$

Note: the region in the xz-plane is not part of the volume!

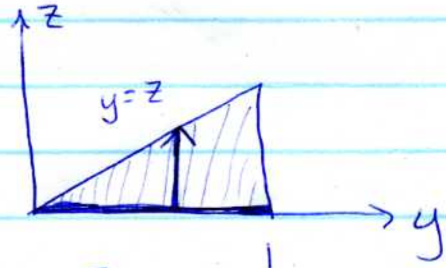




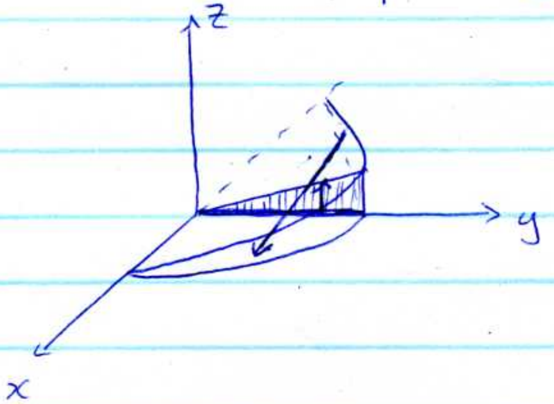
$$E = \{(x,y,z) \mid 0 \le z \le 1,\ -\sqrt{1-z} \le x \le +\sqrt{1-z},\ z \le y \le 1-x^2\}$$



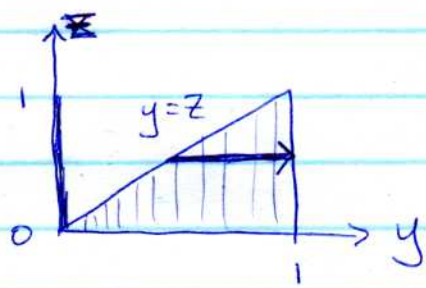
Project onto yz-plane



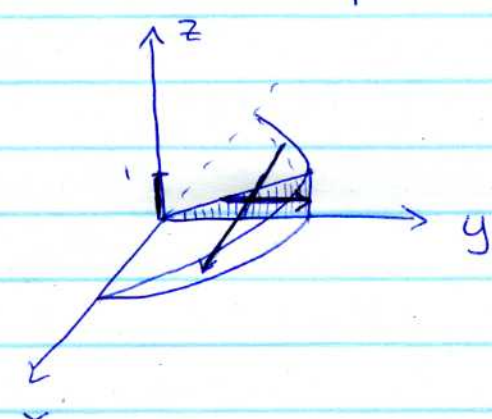
$$E = \{(x,y,z) \mid 0 \le y \le 1,\ 0 \le z \le y,\ -\sqrt{1-y} \le x \le \sqrt{1-y}\}$$



$$\iiint_E dV = \int_0^1 \int_0^y \int_{-\sqrt{1-y}}^{+\sqrt{1-y}} dx\,dz\,dy$$



$$E = \{(x,y,z) \mid 0 \le z \le 1,\ z \le y \le 1,\ -\sqrt{1-y} \le x \le \sqrt{1-y}\}$$

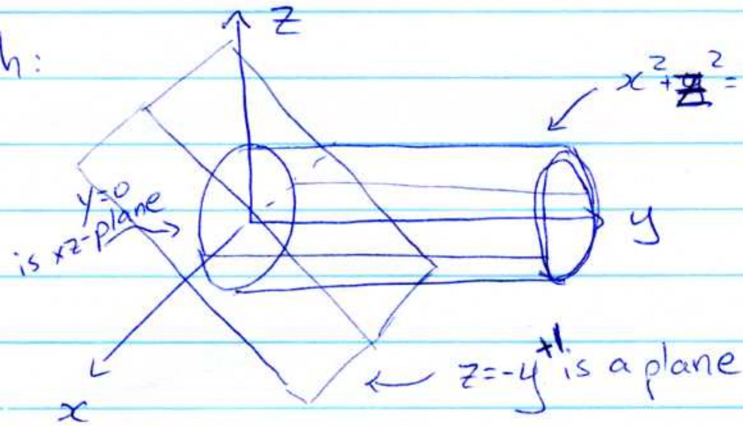


$$\iiint_E dv = \int_0^1 \int_z^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} dx\,dy\,dz$$

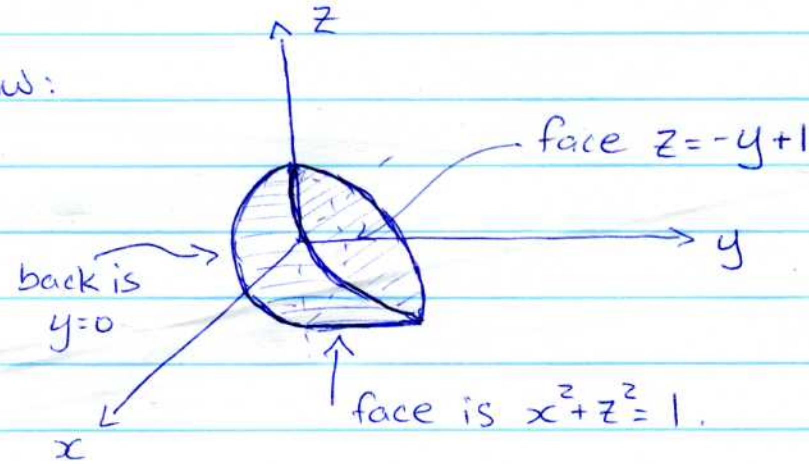
All six integral representations evaluate to $\frac{8}{15}$.

Ex Evaluate $\iiint_D dV$ where D is bounded by $x^2+z^2=1$, $z=-y+1$, $y=0$.

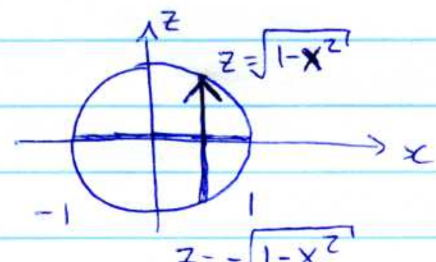
sketch:



Redraw:



project into xz plane:

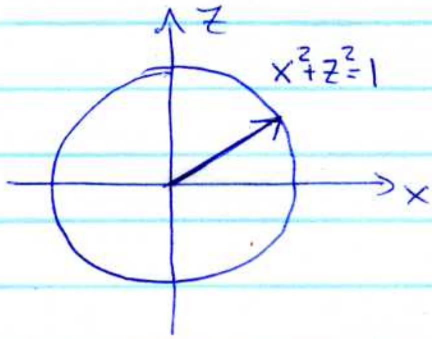


$$D=\left\{(x,y,z) \mid -1\le x\le 1,\ -\sqrt{1-x^2}\le z\le \sqrt{1-x^2},\ 0\le y\le 1-z\right\}$$

$$\iiint_D dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-z} dy\,dz\,dx$$

$$\iiint_D dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-z)\, dz\, dx$$

Seeing those square roots tellsme we might endup with something tricky to evaluate down the road. Let's switch to polar, and see what happens.



$R = \{ (r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1 \}$

$z = r\sin\theta$

$dA = dz\,dx = r\,dr\,d\theta$

$$\iiint_D dV = \int_0^1 \int_0^{2\pi} (1 - r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_0^1 \left(\theta r + r^2\cos\theta \right)\Big|_{\theta=0}^{\theta=2\pi} dr$$

$$= \int_0^1 \left(2\pi r + r^2 - 0 - r^2 \right) dr$$

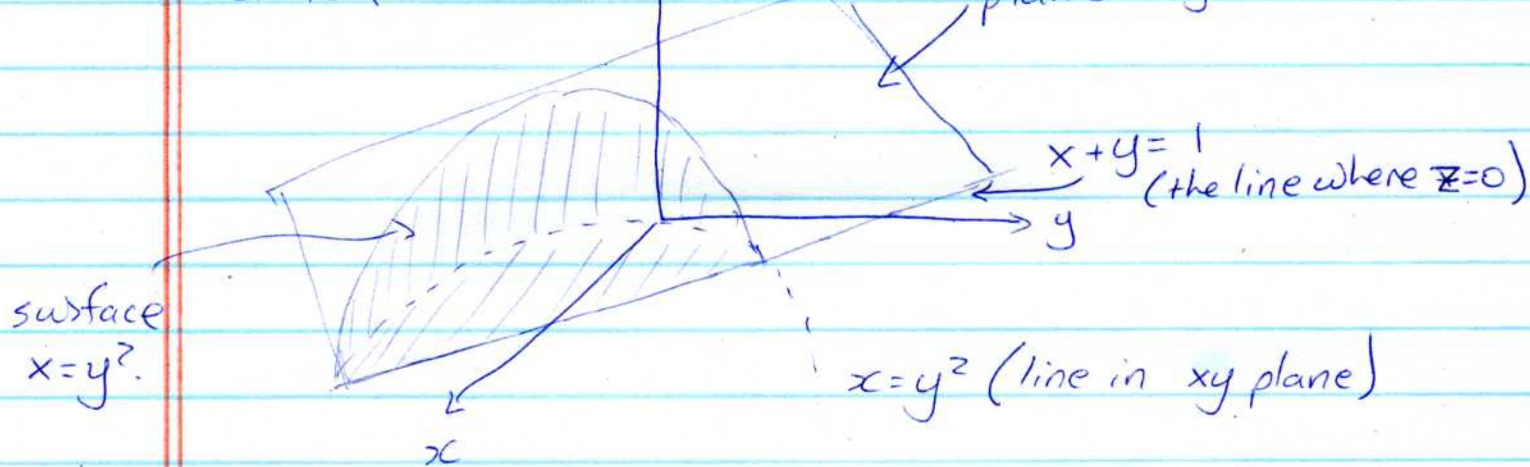
$$= \pi r^2 \Big|_0^1$$

$$= \pi.$$

That was easy.

Ex Find volume of solid bounded by $x = y^2$ and planes $z = 0$ and $x + y + z = 1$.

sketch:



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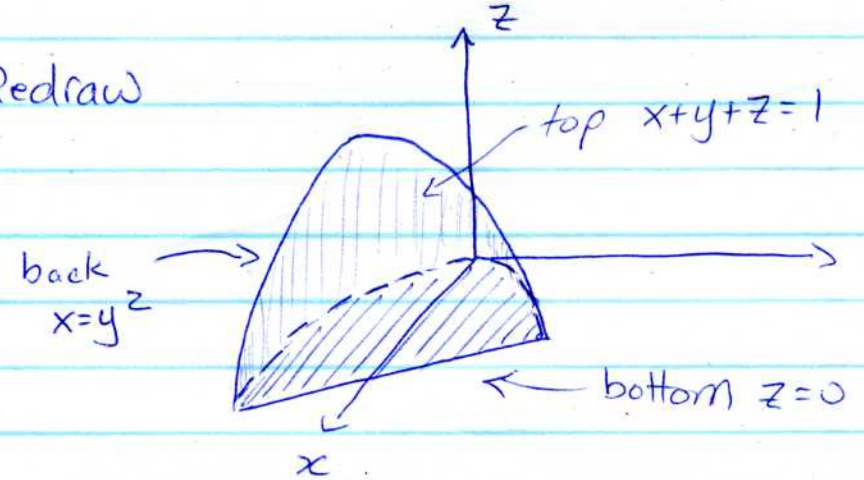
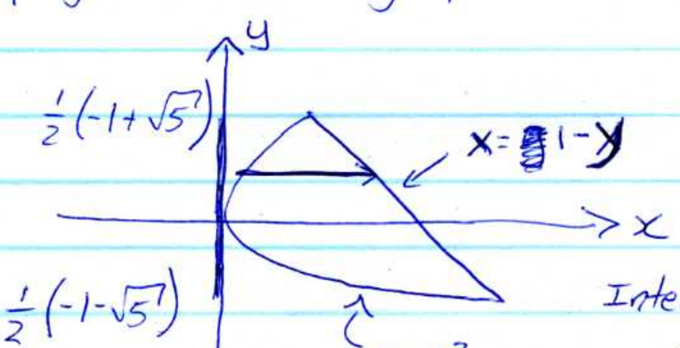


projection in xy plane:



Region $= \{ (x,y,z) \mid$
$\frac{1}{2}(-1-\sqrt{5}) \le y \le \frac{1}{2}(-1+\sqrt{5})$,
$y^2 \le x \le 1-y$,
$0 \le z \le 1-x-y \}$

Intersection:
$y^2 = 1-y$
$\Rightarrow y = \frac{1}{2}(-1 \pm \sqrt{5})$

$$V = \int_{\frac{1}{2}(-1-\sqrt{5})}^{\frac{1}{2}(-1+\sqrt{5})} \int_{y^2}^{1-y} \int_{0}^{1-x-y} dz\,dx\,dy$$

$$= \int_{\frac{1}{2}(-1-\sqrt{5})}^{\frac{1}{2}(-1+\sqrt{5})} \int_{y^2}^{1-y} (1-x-y)\,dx\,dy$$

$$= \int_{\frac{1}{2}(-1-\sqrt{5})}^{\frac{1}{2}(-1+\sqrt{5})} \left(x - \frac{x^2}{2} - yx \right)\Big|_{y^2}^{1-y} dy$$

$$= \int_{\frac{1}{2}(-1-\sqrt{5})}^{\frac{1}{2}(-1+\sqrt{5})} \left(\frac{1}{2} - y - \frac{y^2}{2} + y^3 + \frac{y^4}{2} \right) dy$$

$$= \frac{5\sqrt{5}}{12}$$