Math 2101: Calculus III Vector Spaces

In Section 12.2 the text talks about *properties of vector operations*. These properties are actually the requirement for a mathematical structure called a *vector space*, which is studied in linear algebra. This is an important idea, and so I want you to see it here as well.

The Vector Space Structure

A vector space consists of a set V of objects together with 2 operations on these objects, and these operations satisfy a certain set of rules.

Definition Let V be a set with two operations $\oplus$ and $\odot$. V is called a real vector space, denoted $(V, \oplus, \odot)$, if the following properties hold:

1. vector addition: if $u \in V$ and $v \in V$, then $u \oplus v \in V$ (closure of $\oplus$).
   (a) $v \oplus w = w \oplus v$ for every $v, w, \in V$ (commutativity of $\oplus$)
   (b) $(v \oplus w) \oplus u = v \oplus (w \oplus u)$ for every $u, v, w, \in V$ (associativity of $\oplus$)
   (c) There exists $z \in V$ such that $v \oplus z = z \oplus v = v$ for every $v \in V$ (identity of $\oplus$)
   (d) For every $v \in V$ there exists $w \in V$ such that $v \oplus w = w \oplus v = z$ (w is the negative (or "$\oplus$ inverse") of v)
2. scalar multiplication: if $v \in V$ and $k \in \mathbb{R}$, then $k \odot v \in V$ (closure of $\odot$).
   (a) $k \odot (v \oplus w) = (k \odot v) \oplus (k \odot w)$ for every $u, v \in V$ and $k \in \mathbb{R}$ (distributivity over $\oplus$)
   (b) $(k + j) \odot v = (k \odot v) \oplus (j \odot v)$ for every $v \in V$ and $k, j \in \mathbb{R}$ (distributivity over addition)
   (c) $k \odot (j \odot v) = (kj) \odot v$ for every $v \in V$ and $k, j \in \mathbb{R}$
   (d) $1 \odot v = v$ for every $v \in V$ (scalar multiplication has an identity)

The above ten condition define a vector space. The text lists 8 of these properties simply as properties of vectors, but a vector space can represent much more. The text also has a ninth property ($0\mathbf{u} = \mathbf{0}$) which can be derived from the above ten properties.

Terminology

- The members of the set V are called vectors.
- The real numbers, $\mathbb{R}$, are called scalars.
- The operation $\oplus$ is called vector addition.
- The operation $\odot$ is called scalar multiplication.
- $(V, \oplus, \odot)$ is called a real vector space as we have defined it.
- $(V, \oplus, \odot)$ is a complex vector space if we replace $\mathbb{R}$ by $\mathbb{C}$ (complex numbers).

This is utterly beautiful. It is utterly beautiful since the structure holds for different sets V with different operations $\oplus$ and $\odot$. All the above properties hold for the different vector spaces! Some examples are listed on the next page.

Vector spaces can be many things, including:

$$\begin{aligned}
\text{Matrices: } V &: m \times n \text{ matrices, with } \mathbb{R} \text{ or } \mathbb{C} \text{ elements} \\
\oplus &: \text{matrix addition} \\
\odot &: \text{multiply a matrix by a real number}
\end{aligned}$$

$$\begin{aligned}
\text{Polynomials: } V &: P_n \text{ polynomials of degree } n \text{ or less with } \mathbb{R} \text{ coefficients} \\
\oplus &: \text{adding coefficients of like power} \\
\odot &: \text{multiply each coefficient by a real number}
\end{aligned}$$

$$\begin{aligned}
\text{Coordinate Space } \mathbb{R}^3\text{: } V &: \text{ordered triples } (x_1, x_2, x_3) \\
\oplus &: \text{add components} \\
\odot &: \text{multiply every component by a real number}
\end{aligned}$$

$$\begin{aligned}
\text{Vectors: } V &: n\text{-vectors, } \langle x_1, x_2, x_3, \ldots, x_n \rangle \\
\oplus &: \text{add two vectors} \\
\odot &: \text{multiply a vector by a real number}
\end{aligned}$$

The mathematics underlying all these very different systems is the same–they are all *vector spaces*. This is studied in linear algebra and abstract algebra courses. It is very useful in many fields of study.

Vector Algebra Operations

What are interested in with Section 12.2 uses the following:

$$\begin{aligned}
\text{vectors: } & \mathbf{v} = \langle v_1, v_2, v_3 \rangle \text{ an ordered 3-tuple, with } v_1, v_2, v_3 \in \mathbb{R} \\
\text{scalars: } & k \in \mathbb{R} \\
\text{vector addition } \oplus : & \mathbf{u} + \mathbf{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle \\
\text{scalar multiplication } \oplus : & k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle \text{ with } k \in \mathbb{R}
\end{aligned}$$

and our vector space properties simplify to the following, where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors and k, j are scalars:

1. vector addition: $\mathbf{u} + \mathbf{v}$ is a vector (closure of vector addition).
   (a) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (commutativity of vector addition)
   (b) $(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$ (associativity of vector addition)
   (c) $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ ($\mathbf{0}$ is identity of vector addition)
   (d) $\mathbf{v} - \mathbf{v} = -\mathbf{v} + \mathbf{v} = \mathbf{0}$ ($-\mathbf{v}$ is the negative of $\mathbf{v}$)
2. scalar multiplication: $k\mathbf{v}$ is a vector (closure of scalar multiplication).
   (a) $k(\mathbf{v} + \mathbf{w}) = k\mathbf{v} + k\mathbf{w}$ (distributivity over vector addition)
   (b) $(k + j)\mathbf{v} = k\mathbf{v} + j\mathbf{v}$) (distributivity over addition)
   (c) $k(j\mathbf{v}) = (kj)\mathbf{v}$
   (d) $1\mathbf{v} = \mathbf{v}$ (1 is identity for scalar multiplication)