Questions

(30) 1. In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution of

$$y'' + y = \begin{cases} t & 0 \le t \le \pi \\ \pi e^{\pi - t} & t > \pi \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are continuous at $t = \pi$.

Plot the nonhomogeneous term and the solution as a function of time.

Hint: First solve the IVP for $t \le \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity and differentiability conditions at $t = \pi$.

To get the sketches asked for, use *Mathematica* and the intrinsic command **Piecewise**.

For example, to define the function

$$g(t) = \begin{cases} \sin(t + \pi/2) & t < 0 \\ t^2 + 1 & t \ge 0 \end{cases}$$

use `g[t_]=Piecewise[{{Sin[t+Pi/2],t<0},{t^2+1,t>=0}}]`

This problem gives you practice with the undetermined coefficients technique.

(30) 2. A *hardening spring* is a spring that does not satisfy Hooke's law, but instead the restoring force satisfies $F_s = -(ku + \epsilon u^3)$ where $\epsilon > 0$ is small (if $\epsilon < 0$ it is called a softening spring). The effect of a hardening spring is that the restoring force is stronger for larger displacements than for a spring that satisfies Hooke's law.

The differential equation this spring satisfies is $mu'' + \gamma u' + ku + \epsilon u^3 = F(t)$.

Since this is nonlinear, our techniques of solution will not work. Analytic solutions are typically based around a perturbation method which exploits the fact that ϵ is small. If you want to see more about the perturbation techniques, you can check out the following article:
http://www.ewp.rpi.edu/hartford/ ernesto/F2011/EP/MaterialsforStudents/Royle/Kimiaeifar2009.PDF

We will instead investigate using numerical solutions. Consider the IVP:

$$u'' + \frac{1}{5}u' + u + \frac{1}{5}u^3 = \cos(\omega t), \;\; u(0) = 0, \;\; u'(0) = 0.$$

(a) Using *Mathematica*'s command `NDSolve`, generate plots of the solution for several values of ω between 1/2 and 2 and estimate the amplitude R of the steady-state response in each case (you can do this visually from a graph for a variety of ω).

(b) Using the data from part (a), plot the graph of R versus ω. For what frequency is the amplitude the greatest?

(c) Compare the results in a and b with the results for the linear spring ($\epsilon = 0$, see Figure 3.8.2 in the text and read page 209 to create the graph like Fig 3.8.2 for this particular case).

This problem gives you practice with Mathematica's numerical solver. We will develop a way later in the semester to solve problems like this one using a modification of Taylor's Method of Order n.

(20) 3. Find the real valued general solution to $y^{(6)} + y = 0$. Use the ideas from the roots of unity discussion to help you factor the characteristic equation (I want to see the details of this computation, including a sketch of the roots in the complex plane).

This problem shows you one example of how roots of polynomials are related to differential equations.

(20) 4. Consider the initial value problem: $4y^{(3)} + 4y^{(2)} + y^{(1)} + \beta y = 0, \;\; y(0) = 0, y'(0) = 1, y''(0) = 3.$

Solve and plot the IVP for $\beta = \frac{2}{27}, \frac{2}{27} - \frac{1}{40}, \frac{2}{27} + \frac{1}{40}$. What is special about the characteristic equation near $\beta = \frac{2}{27}$?

You may use *Mathematica*'s `DSolve` command on this problem.

This problem shows the sensitivity of the solution to the value of β when $\beta \sim \frac{2}{27}$. This differential equation is also sensitive for $\beta \sim 0$.