Questions

(20) 1. Find the recurrence relation for the power series solution about $x = 0$ for the initial value problem

$$(1-x)y'' + xy' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Then, use *Mathematica* to plot the solution when you keep 20 and 40 terms in the series.

This problem lets you see what happens when the series is truncated.

(20) 2. Plot several partial sums in a series solution of the given initial value problem about $x = 0$, thereby obtaining graphs analogous to those in Figures 5.2.1 through 5.2.4.

$$y'' + x^2 y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

This problem lets you see what happens when the series is truncated.

(20) 3. Find the fundamental set of solutions to the differential equation

$$y'' + x^2 y' + xy = 0$$

that arises with a series solution about $x = 0$.

(a) You will want to find the pattern in the recursion relation.

(b) Use *Mathematica* to determine what underlying functions the two series solutions represent (sum out to infinity).

(c) Use *Mathematica* to plot the solution to the IVP

$$y'' + x^2 y' + xy = 0, \quad y(0) = 7, \; y'(0) = 10.$$

(d) Finally, use *Mathematica*'s built-in command `DSolve` to solve the IVP and compare the plot of this solution to the one you found earlier.

This gives you practice searching for patterns, and using Mathematica to assist in solutions.

(20) 4. By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation. For example, show the solution of

$$x^2 y'' + (\alpha^2 \beta^2 x^{2\beta} + \frac{1}{4} - v^2 \beta^2) y = 0, \; x > 0$$

is given by $y = \sqrt{x} f(\alpha x^\beta)$ where $f = J_v$ is the solution of the Bessel equation of order v.

This problem gives you a chance to learn a little bit more about how to convert differential equation into other differential equations by changing variables. The process is similar in spirit to what we did in class to solve the Euler equation. You can begin by making a change of variables $t = \alpha x^\beta$, and then converting the given differential equation in x and y into the Bessel DE in t and f. This is a useful idea. You can do this entirely by hand, or Mathematica can assist with some of the derivatives.

(20) 5. Let $f(t+T) = f(t)$ for all $t \geq 0$ where $T > 0$ is a constant. Show that the Laplace transform satisfies the formula

$$\mathcal{L}[f(t)] = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$$

Hint: Try to expand the integral in a series that allows you to exploit the periodicity of f, then make a substitution in the integrals to make the integrals the same and use the geometric series to simplify.

This is an important theoretical result necessary for working with periodic functions and Laplace transforms.