Math 2401: DE Assignment 4

(25) 1. Determine the solution to the IVP $y'' + y = \sum_{k=1}^{\infty} k\,\delta(t-k)$, $y(0) = y'(0) = 1$ using Laplace transforms and the table of Laplace transforms. Then use *Mathematica* to plot the solution for $0 < t < 20$.

The function $\sum_{k=-\infty}^{\infty} \delta(t-k)$ *is known as the Dirac Comb (we've started at* $k = 1$ *here rather than* $k = -\infty$ *and modified it slightly) which often appears in electrical engineering problems. It is a train of equally spaced impulse functions. This type of problem is one that Laplace transforms handle especially well. Imagine what would have to be done to solve this without Laplace transforms!*

(25) 2. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} \mathbf{x}.$$

a) Find the general solution to the system and describe the behaviour of the solution as $t \to \infty$.

b) Check that the two solutions you find are linearly independent.

c) Use *Mathematica* to create a direction field for the system and plot a few trajectories of the system.

This problem gives you practice with the basic method of solving a system of differential equations by solving the associated eigensystem problem, and the new method of calculating a Wronskian.

(25) 3. Find the general <u>real-valued solution</u> to

$$t\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix} \mathbf{x}, \qquad t > 0.$$

Hint: The system $t\mathbf{x}' = \mathbf{A}\mathbf{x}$ is analogous to the Euler equation. Your assumed solution should be suitably modified. You do not have to verify that your solutions form a fundamental set of solutions.

Solving this problem shows you how the concept of an Euler equation extends to a system of differential equations, as well as how to construct real valued solutions from complex solutions.

(25) 4. By computing eigenvalues and eigenvectors, and using undetermined coefficients, solve

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} t.$$

Solving this problem gives you a chance to explore undetermined coefficients applied to a nonhomogeneous system of differential equations.

Solutions

Problem 1. Let's take the Laplace transform of the differential equation:

$$y'' + y = \sum_{k=1}^{\infty} k\delta(t-k)$$

$$\mathcal{L}[y''] + \mathcal{L}[y] = \sum_{k=1}^{\infty} k\mathcal{L}[\delta(t-k)]$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \sum_{k=1}^{\infty} ke^{-ks} \text{ Using Table 6.2.1 \#18 and \#17}$$

$$s^2Y(s) - s - 1 + Y(s) = \sum_{k=1}^{\infty} ke^{-ks}$$

$$Y(s)(s^2+1) = s + 1 + \sum_{k=0}^{\infty} ke^{-ks}$$

$$Y(s) = \frac{s+1}{s^2+1} + \sum_{k=1}^{\infty} ke^{-ks} \cdot \frac{1}{s^2+1}$$

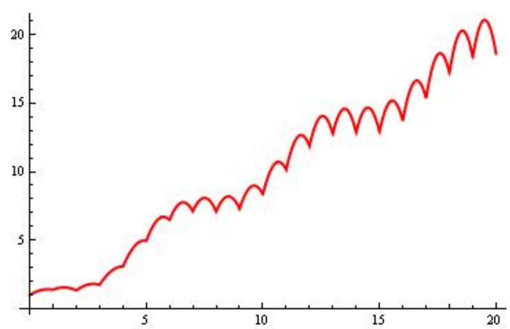
$$\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{s+1}{s^2+1}\right] + \sum_{k=1}^{\infty} k\mathcal{L}^{-1}\left[e^{-ks} \cdot \frac{1}{s^2+1}\right]$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] + \sum_{k=1}^{\infty} ku_k(t)\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]_{t-k}$$

$$y(t) = \cos t + \sin t + \sum_{k=1}^{\infty} ku_k(t)\sin(t-k)$$

Since we want a plot out to $t = 20$, we can truncate our Dirac comb at $t = 20$.

```
ysol[t_] = Cos[t] + Sin[t] + Sum[k*HeavisideTheta[t - k]*Sin[t - k], {k, 1, 20}]
plot2 = Plot[ysol[t], {t, 0, 20}, PlotStyle -> {Red, Thick}]
```



Problem 2. Assume that $\mathbf{x} = \xi e^{\lambda t}$ is a solution, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{aligned} & \mathbf{x} = \xi e^{\lambda t} \\ & \mathbf{x}' = \lambda \xi e^{\lambda t} \\ \text{Substitute:} \quad & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\ e^{\lambda t} \neq 0: \quad & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \\ & \begin{pmatrix} 5/4 - \lambda & 3/4 \\ 3/4 & 5/4 - \lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Eigenvalues:

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= 0 \\ (5/4 - \lambda)(5/4 - \lambda) - 9/16 &= 0 \\ 25/16 - 5\lambda/2 + \lambda^2 - 9/16 &= 0 \\ \lambda^2 - 5\lambda/2 + 1 &= 0 \\ (\lambda - 2)(2\lambda - 1) &= 0 \end{aligned}$$

The eigenvalues are $\lambda^{(1)} = +2$ and $\lambda^{(2)} = 1/2$.

Eigenvectors:

$\lambda^{(1)} = 2$:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)} I)\xi^{(1)} &= \mathbf{0} \\ (\mathbf{A} - 2I)\xi^{(1)} &= \mathbf{0} \\ \begin{pmatrix} 5/4 - 2 & 3/4 \\ 3/4 & 5/4 - 2 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -3/4 & 3/4 \\ -3/4 & 3/4 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} -3/4\xi_1^{(1)} + 3/4\xi_2^{(1)} &= 0 \\ -3/4\xi_1^{(1)} + 3/4\xi_2^{(1)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(1)}$ to be arbitrary. Set $\xi_2^{(1)} = 1$.
Therefore, $\xi_1^{(1)} = \xi_2^{(1)} = 1$.

$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = 2$.

A solution of the system of differential equations is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$.

$\underline{\lambda^{(2)} = 1/2}$:

$$\begin{aligned}
(\mathbf{A} - \lambda^{(2)} I)\xi^{(2)} &= \mathbf{0} \\
(\mathbf{A} + I)\xi^{(2)} &= \mathbf{0} \\
\begin{pmatrix} 5/4 - 1/2 & 3/4 \\ 3/4 & 5/4 - 1/2 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 3/4 & 3/4 \\ 3/4 & 3/4 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

So we get the two equations:

$$\begin{aligned}
3/4\xi_1^{(2)} + 3/4\xi_2^{(2)} &= 0 \\
3/4\xi_1^{(2)} + 3/4\xi_2^{(2)} &= 0
\end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(2)}$ to be arbitrary. Set $\xi_2^{(2)} = 1$.
Therefore, $\xi_1^{(2)} = -\xi_2^{(2)} = -1$.

$\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(2)} = 1/2$.

A solution of the system of differential equations is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}$.

<u>The General Solution</u>:
Check linear independence by computing the Wronskian:

$$W(t) = \det \Psi(t) = \det \left(\mathbf{x}^{(1)} \mathbf{x}^{(2)} \right) = \begin{vmatrix} e^{2t} & e^{t/2} \\ e^{2t} & -e^{t/2} \end{vmatrix} = -2e^{5t/2} \neq 0$$

Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent. Therefore, they form a fundamental set of solutions.

A general solution is therefore

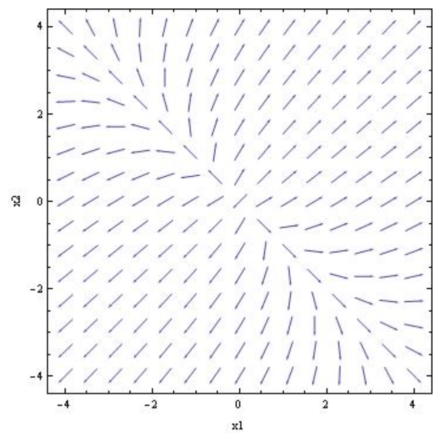
$$\begin{aligned}
\mathbf{x} &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} \\
&= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t/2}
\end{aligned}$$

The general solution could also be written in terms of the fundamental matrix and a constant vector $\mathbf{c}$. This constant vector represents the constants of integration.

$$\mathbf{x} = \begin{pmatrix} e^{2t} & e^{t/2} \\ e^{2t} & -e^{t/2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Psi(t)\mathbf{c}$$

Here are the *Mathematica* commands I used, as well as the output graphics:

```
field=VectorPlot[{5/4x1+3/4 x2,3/4 x1+5/4 x2},{x1,-4,4},{x2,-4,4},Frame->True,
FrameLabel->{"x1","x2"},VectorScale->{Tiny,Tiny,None}]
```



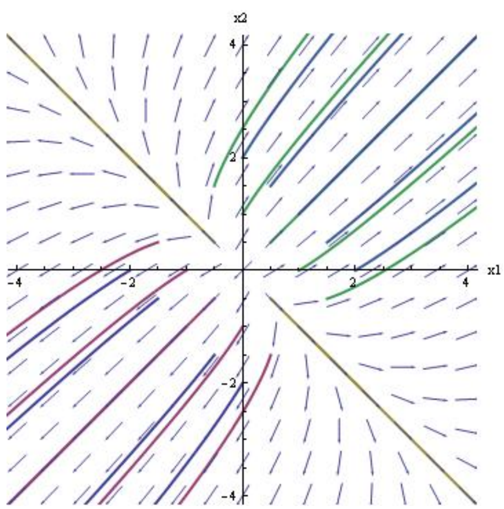
Solution and Direction Field:

```
x1 = c1  Exp[2t] + c2 Exp[t/2]
x2 = c1  Exp[2t] - c2 Exp[t/2]

functionlist=Table[{x1,x2},{c1,-1,1,0.5},{c2,-1,1,0.5}]

plot1 =ParametricPlot[functionlist,{t,0,3},AxesLabel->{"x1","x2"},PlotStyle->{Thick}];

Show[plot1,field,PlotRange->{{-4,4},{-4,4}},AxesLabel->{"x1","x2"},AspectRatio->1,AxesOrigin->{0,0}]
```



As $t \to \infty$, the solution approaches $(x_1, x_2) = (\pm\infty, \pm\infty)$, depending on where you start you get $\pm\infty$.

Problem 3. Assume that $\mathbf{x} = \xi t^{\lambda}$ is a solution, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{aligned}
& \mathbf{x} = \xi t^{\lambda} \\
& \mathbf{x}' = \lambda \xi t^{\lambda-1} \\
\text{Substitute:} \quad & t\lambda\xi t^{\lambda-1} = \mathbf{A}\xi t^{\lambda} \\
& \lambda\xi t^{\lambda} = \mathbf{A}\xi t^{\lambda} \\
t^{\lambda} \neq 0 (t > 0): \quad & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \\
& \begin{pmatrix} -1-\lambda & -1 \\ 4 & -1-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

Eigenvalues:

$$\begin{aligned}
\det(\mathbf{A} - \lambda I) &= 0 \\
(-1-\lambda)(-1-\lambda) + 4 &= 0 \\
(-1-\lambda)^2 &= -4 \\
-1-\lambda &= \pm 2i \\
\lambda &= -1 \pm 2i
\end{aligned}$$

The eigenvalues are $\lambda^{(1)} = -1 + 2i$ and $\lambda^{(2)} = -1 - 2i$.

Eigenvectors:

$\lambda^{(1)} = -1 + 2i$:

$$\begin{aligned}
(\mathbf{A} - \lambda^{(1)} I)\xi^{(1)} &= \mathbf{0} \\
(\mathbf{A} - (-1+2i)I)\xi^{(1)} &= \mathbf{0} \\
\begin{pmatrix} -1+1-2i & -1 \\ 4 & -1+1-2i \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

So we get the two equations:

$$\begin{aligned}
-2i\xi_1^{(1)} - \xi_2^{(1)} &= 0 \\
4\xi_1^{(1)} - 2i\xi_2^{(1)} &= 0
\end{aligned}$$

These are the same equation (multiply the first by $2i$ to see this). So we have 1 equation with 2 unknowns.
Choose $\xi_1^{(1)}$ to be arbitrary. Set $\xi_1^{(1)} = 1$.
Therefore, $\xi_2^{(1)} = -2i\xi_2^{(1)} = -2i$.

$\xi^{(1)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = -1 + 2i$.

A solution of the system of differential equations is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -2i \end{pmatrix} t^{(-1+2i)}$.

Since the matrix was real valued, we know that the eigenvalues and eigenvectors will occur in complex conjugate pairs. This means we know a second solution will be $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2i \end{pmatrix} t^{(-1-2i)}$.

To get real valued solutions, we can split $\mathbf{x}^{(1)}$ into real and complex parts. Each of these will be a solution, and each will be real valued.

$$\begin{aligned}
\mathbf{x}^{(1)} &= \begin{pmatrix} 1 \\ -2i \end{pmatrix} t^{(-1+2i)} \\
&= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} i\right] t^{-1} e^{2i\ln t} \\
&= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} i\right] t^{-1} \left[\cos(2\ln t) + i\sin(2\ln t)\right] \\
&= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-1}\cos(2\ln t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} t^{-1}\sin(2\ln t)\right] + i\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-1}\sin(2\ln t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} t^{-1}\cos(2\ln t)\right] \\
&= \mathbf{u}(t) + i\mathbf{v}(t)
\end{aligned}$$

Two real valued solutions are $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

$$\begin{aligned}
\mathbf{u}(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-1}\cos(2\ln t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} t^{-1}\sin(2\ln t) = \begin{pmatrix} t^{-1}\cos(2\ln t) \\ 2t^{-1}\sin(2\ln t) \end{pmatrix} \\
\mathbf{v}(t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^{-1}\sin(2\ln t) - \begin{pmatrix} 0 \\ 2 \end{pmatrix} t^{-1}\cos(2\ln t) = \begin{pmatrix} t^{-1}\sin(2\ln t) \\ -2t^{-1}\cos(2\ln t) \end{pmatrix}
\end{aligned}$$

A general solution is therefore

$$\begin{aligned}
\mathbf{x} &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} \\
&= c_1\begin{pmatrix} t^{-1}\cos(2\ln t) \\ 2t^{-1}\sin(2\ln t) \end{pmatrix} + c_2\begin{pmatrix} t^{-1}\sin(2\ln t) \\ -2t^{-1}\cos(2\ln t) \end{pmatrix}
\end{aligned}$$

Problem 4. First, solve the associated homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Assume that $\mathbf{x} = \xi e^{\lambda t}$ is a solution, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{aligned}
& \mathbf{x} = \xi e^{\lambda t} \\
& \mathbf{x}' = \lambda \xi e^{\lambda t} \\
\text{Substitute:}\quad & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\
e^{\lambda t} \neq 0: \quad & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \\
& \begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

<u>Eigenvalues</u>:

$$\begin{aligned}
\det(\mathbf{A} - \lambda I) &= 0 \\
(-2-\lambda)(-2-\lambda) - 1 &= 0 \\
(-2-\lambda)^2 &= 1 \\
-2-\lambda &= \pm 1 \\
\lambda &= -2 \pm 1
\end{aligned}$$

The eigenvalues are $\lambda^{(1)} = -1$ and $\lambda^{(2)} = -3$.

<u>Eigenvectors</u>:

<u>$\lambda^{(1)} = -1$</u>:

$$\begin{aligned}
(\mathbf{A} - \lambda^{(1)} I)\xi^{(1)} &= \mathbf{0} \\
(\mathbf{A} + I)\xi^{(1)} &= \mathbf{0} \\
\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

So we get the two equations:

$$\begin{aligned}
-\xi_1^{(1)} + \xi_2^{(1)} &= 0 \\
\xi_1^{(1)} - \xi_2^{(1)} &= 0
\end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(1)}$ to be arbitrary. Set $\xi_2^{(1)} = 1$.
Therefore, $\xi_1^{(1)} = \xi_2^{(1)} = 1$.

$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = -1$.

A solution of the system of differential equations is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$.

$\underline{\lambda^{(2)} = -3}$:

$$\begin{aligned}
(\mathbf{A} - \lambda^{(2)} I)\xi^{(2)} &= \mathbf{0} \\
(\mathbf{A} + 3I)\xi^{(2)} &= \mathbf{0} \\
\begin{pmatrix} -2+3 & 1 \\ 1 & -2+3 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

So we get the two equations:

$$\begin{aligned}
\xi_1^{(2)} + \xi_2^{(2)} &= 0 \\
\xi_1^{(2)} + \xi_2^{(2)} &= 0
\end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(2)}$ to be arbitrary. Set $\xi_2^{(2)} = 1$.
Therefore, $\xi_1^{(2)} = -\xi_2^{(2)} = -1$.

$\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(2)} = -3$.

A solution of the system of differential equations is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$.

Check linear independence by computing the Wronskian:

$$W(t) = \det \Psi(t) = \det \left(\mathbf{x}^{(1)}\mathbf{x}^{(2)}\right) = \begin{vmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{vmatrix} = -e^{-4t} - e^{-4t} = -2e^{-4t} \neq 0$$

Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent. Therefore, they form a fundamental set of solutions.

The complementary solution is therefore

$$\begin{aligned}
\mathbf{x} &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} \\
&= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}
\end{aligned}$$

Now we use undetermined coefficients to determine a particular solution of the nonhomogeneous equation.

First, examine the nonhomogeneous term $\mathbf{g}(t)$:

$$\mathbf{g}(t) = \begin{pmatrix} -4 \\ 1 \end{pmatrix} t$$

Since this is a polynomial we assume a solution looks like:

$$\mathbf{x} = \mathbf{v} = \mathbf{a}t + \mathbf{b}$$

where $\mathbf{a}, \mathbf{b}$ are 2 vectors. Is any part of this included in any part of the complementary solution? No. Therefore, we know this assumed form of the solution will work.

Differentiate and substitute into the system of differential equations:

$$\begin{aligned}
\mathbf{v} &= \mathbf{a}t + \mathbf{b} \\
\mathbf{v}' &= \mathbf{a} \\
\mathbf{a} &= \mathbf{A}(\mathbf{a}t + \mathbf{b}) + \begin{pmatrix} -4 \\ 1 \end{pmatrix} t
\end{aligned}$$

Collect terms:

$$(\mathbf{A}\mathbf{b} - \mathbf{a}) + t\left(\mathbf{A}\mathbf{a} + \begin{pmatrix} -4 \\ 1 \end{pmatrix}\right) = \mathbf{0}$$

And we get the two equations in the two unknowns $\mathbf{a}, \mathbf{b}$ which must be satisfied (setting coefficients of powers of t to zero will make the solution satisfy the differential equation for all values of t):

$$\mathbf{A}\mathbf{b} - \mathbf{a} = \mathbf{0} \tag{1}$$

$$\mathbf{A}\mathbf{a} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \mathbf{0} \tag{2}$$

Solve the second equation for $\mathbf{a}$ using Cramer's Rule:

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$a_1 = \frac{\begin{vmatrix} 4 & 1 \\ -1 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-8+1}{3} = -\frac{7}{3} \qquad a_2 = \frac{\begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{2-4}{3} = -\frac{2}{3}$$

Therefore $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -7/3 \\ -2/3 \end{pmatrix}$.

The first equation now becomes

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -7/3 \\ -2/3 \end{pmatrix}$$

$$b_1 = \frac{\begin{vmatrix} -7/3 & 1 \\ -2/3 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{14/3 + 2/3}{3} = \frac{16}{9} \qquad b_2 = \frac{\begin{vmatrix} -2 & -7/3 \\ 1 & -2/3 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{4/3 + 7/3}{3} = \frac{11}{9}$$

Therefore $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 16/9 \\ 11/9 \end{pmatrix}$.

A particular solution to the nonhomogeneous equation is

$$\mathbf{x}_p = \begin{pmatrix} -7/3 \\ -2/3 \end{pmatrix} t + \begin{pmatrix} 16/9 \\ 11/9 \end{pmatrix}$$

The general solution of the nonhomogeneous equation is therefore

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} -7/3 \\ -2/3 \end{pmatrix} t + \begin{pmatrix} 16/9 \\ 11/9 \end{pmatrix}.$$