Math 2401 Differential Equations Test 2
Wed Oct 31, 2012 (65 minutes)
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Instructions:

- Attempt all questions.
- The test is out of 100 marks.
- There are 5 questions, 20 marks each.
- You have 65 minutes to complete the test.
- You may use a calculator if you desire.
- Budget your time.
- Do questions which you know how to do immediately first.
- Leave questions which you find difficult until last.
- Ask for clarification if you do not understand a question.
- You must show your work. Label sketches well.

Problem 1. (20 marks) True or False (2 marks each): (i)–(v)

(i) If the characteristic equation associated with a fourth order linear differential equation has one real root r of multiplicity four, then a fundamental set of solutions is $\{te^{rt}, t^2e^{rt}, t^3e^{rt}, t^4e^{rt}\}$ .. $\{e^{rt}, te^{rt}, t^2e^{rt}, t^3e^{rt}\}$ .. T [F]

(ii) The function $f(x) = \ln x$ is analytic at $x_0 = 1$ .. not analytic at $x=0$ .... [T] F

(iii) The differential equation $x^2y'' + 4xy' + 2y = 0$ has the solution $y = c_1x^{-1} + c_2x^{-2}$ .. Euler .... [T] F

(iv) Without solving for the series, you know the differential equation $(1+x)y'' - xy' - y = 0$ has two linearly independent solutions of the form $y = \sum_{n=0}^{\infty} a_n(x-1)^n$ .. $x=1$ is ordinary point. .... [T] F

(v) The differential equation $2x(x-2)^2y'' + 3xy' + (x-2)y = 0$ has a regular singular point at $x_0 = 2$
.. $p(x) = \frac{3}{2(x-2)^2}$ ...... $q(x) = \frac{1}{2x(x-2)}$ ...... $(x-2)p(x) = \frac{3}{2(x-2)}$ .. not analytic at $x=2$ .... T [F]
→ irregular singular point.

Multiple Choice (2 marks each): For each differential equation given in (vi)–(x), identify the best possible assumed form for the solution from the following choices:

A) $y = e^{rx}$ B) $y = x^r$ C) $y = \sum_{n=0}^{\infty} a_n x^n$ D) $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

(vi)	$\pi x^3 y''' + x^2 y'' - 2xy' + 13y = 0$	A	[B]	C	D	(Euler Equation)
(vii)	$\pi y'' + ey' + \pi^e y = 0$	[A]	B	C	D	(constant coefficient)
(viii)	$xy'' + 17xy' + \pi y = 0$	A	B	C	[D]	($x = 0$ is regular singular point)
(ix)	$21(1-x)y'' + 7xy' + y = 0$	A	B	[C]	D	($x = 0$ is an ordinary point)
(x)	$(y')^2 - y^2 - 1 = 0$	A	B	[C]	D ↑ D would also work.	(nonlinear DE, try a Taylor series)

Problem 2. (20 marks) Find a general solution of $y^{(4)} - 2y^{(2)} + y = t^2$.

Solution First, solve the associated homogeneous equation: $y^{(4)} - 2y^{(2)} + y = 0$.

This is a constant coefficient equation, so we assume $y = e^{rt}$.
Differentiate and substitute into the differential equation:

$$\begin{aligned} y^{(4)} - 2y^{(2)} + y &= 0 \\ (r^4 - 2r^2 + 1)e^{rt} &= 0 \\ r^4 - 2r^2 + 1 &= 0 \quad \text{characteristic equation} \\ r^4 - 2r^2 + 1 = (r^2 - 1)^2 &= 0 \end{aligned}$$

Roots of the characteristic equation are $r = \pm 1$ of multiplicity 2.

A fundamental set of solutions is $y_1 = e^t$, $y_2 = e^{-t}$, $y_3 = te^t$, $y_4 = te^{-t}$.

The complementary solution is $y_c(t) = c_1e^t + c_2e^{-t} + c_3te^t + c_4te^{-t}$.

To get a particular solution we will use undetermined coefficients.
Assume a solution exists of the form: $Y(t) = At^2 + Bt + C$.
This will work, since no part of the assumed solution is contained in the complementary solution.
Differentiate and substitute into the differential equation:

$$\begin{aligned} Y(t) &= At^2 + Bt + C \\ Y^{(1)}(t) &= 2At + B \\ Y^{(2)}(t) &= 2A \\ Y^{(3)}(t) = Y^{(4)}(t) &= 0 \end{aligned}$$

$$\begin{aligned} Y^{(4)} - 2Y^{(2)} + Y &= t^2 \\ 0 - 2(2A) + (At^2 + Bt + C) &= t^2 \\ (-4A + C) + Bt + (A - 1)t^2 &= 0 \end{aligned}$$

For this to be true for all values of t, the coefficients of powers of t must be zero.
This means $A = 1$ and $B = 0$, $C = 4$.

A particular solution is $y_p(t) = t^2 + 4$.

The general solution is $y(t) = y_c(t) + y_p(t) = c_1e^t + c_2e^{-t} + c_3te^t + c_4te^{-t} + t^2 + 4$.

Problem 3. (20 marks) Solve the initial value problem

$$3x^2y'' + xy' - y = 0, \quad y(1) = 1, y'(1) = 1.$$

Solution This is an Euler equation, so the solution should look like $y = x^r$.

Substitute this into the differential equation, using $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$:

$$\begin{aligned}
3x^2y'' + xy' - y &= 0 \\
3x^2r(r-1)x^{r-2} + xrx^{r-1} - x^r &= 0 \\
3r(r-1) + r - 1 &= 0, \text{ cancelled } x^r\text{, which is allowed if } x > 0 \\
3r^2 - 2r - 1 &= 0, \text{ indicial equation} \\
r &= \frac{1}{2a}\Big(-b \pm \sqrt{b^2 - 4ac}\Big) \\
&= \frac{1}{6}\Big(2 \pm \sqrt{4+12}\Big) \\
&= \frac{1}{6}\Big(2 \pm 4\Big) \\
&= 1, -\frac{1}{3}
\end{aligned}$$

So two solutions are $y_1 = x, y_2 = x^{-1/3}$. These form a fundamental set of solutions.

A general solution is $y = c_1x + c_2x^{-1/3}$.

Use the initial conditions to determine the constants c_1 and c_2.

$$y(1) = 1 = c_1 + c_2 \tag{1}$$

$$y'(1) = 1 = c_1 - \frac{1}{3}c_2 \tag{2}$$

The solution to this system of equations can be found using Cramer's rule:

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & -\frac{1}{3} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -\frac{1}{3} \end{vmatrix}} = 1$$

and knowing $c_1 = 1$, we can see immediately from either of Eqs. (1) or (2) that $c_2 = 0$.

The solution to the IVP is therefore given by $y(x) = x$.

Problem 4. (20 marks) Determine the recurrence relation for the series solution about the ordinary point $x_0 = 2$ of the differential equation

$$y'' - xy = 0.$$

You do not have to find the series solutions, nor discuss convergence.

Solution Since the point $x = 2$ is an ordinary point for this differential equation, we assume the following solution:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n (x-2)^n \\
y' &= \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2}
\end{aligned}$$

Since we are working around $x = 2$, we should replace $x = 2 + (x-2)$ in the differential equation. Now, we substitute into the differential equation.

$$\begin{aligned}
y'' - (2 + (x-2))y &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} - (2+(x-2))\sum_{n=0}^{\infty} a_n(x-2)^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=0}^{\infty} a_n(x-2)^{n+1} &= 0 \\
\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n - \sum_{n=0}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{n-1}(x-2)^n &= 0 \\
(2a_2 - 2a_0)(x-2)^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n - \sum_{n=1}^{\infty} 2a_n(x-2)^n - \sum_{n=1}^{\infty} a_{n-1}(x-2)^n &= 0 \\
(2a_2 - 2a_0)(x-2)^0 + \sum_{n=1}^{\infty} \Big((n+2)(n+1)a_{n+2} - 2a_n - a_{n-1}\Big)(x-2)^n &= 0
\end{aligned}$$

If this relation is true for all values of x, then all the coefficients of powers of $x - 1$ must be zero:

$$2a_2 - 2a_0 = 0$$

$$(n+2)(n+1)a_{n+2} - 2a_n - a_{n-1} = 0, \qquad n = 1, 2, 3, 4 \ldots.$$

These are the recurrence relations. They cannot be combined into one, since a_{-1} is not defined.

Problem 5. (20 marks) In solving a differential equation by a series method about $x = 0$, you arrive at the following indicial equation and recurrence relation:

$$r(2r-1) = 0; \qquad (n+r+1)(3n+3r+1)a_{n+1} - a_n = 0, \ n = 0, 1, 2, 3, \ldots$$

Find two solutions in a fundamental set of solutions to the differential equation. Include the first four nonzero terms in each solution. You do not have to discuss convergence, nor find a general term in each series.

Solution The roots of the indicial equation are $r = 0$ and $r = 1/2$. For each of these roots we will get a different recurrence relation, which we can solve since the roots do not differ by an integer.

Solutions will look like
$y \sim x^r \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3\right]$
(exact is $\sum_{n=0}^{\infty} a_n x^{n+r}$).

$r_1 = 1/2$:
The recurrence relation becomes:

$$a_{n+1} = \frac{a_n}{(n+3/2)(3n+5/2)} = \frac{4a_n}{(2n+3)(6n+5)}, n = 0, 1, 2, 3, \ldots$$

$$a_0 = \text{arbitrary}$$

$$a_1 = \frac{4a_0}{3 \cdot 5} = \frac{4a_0}{15} = \frac{4a_0}{3 \cdot 5}$$

$$a_2 = \frac{4a_1}{5 \cdot 11} = \frac{16a_0}{55 \cdot 15} = \frac{16a_0}{825} = \frac{16a_0}{3 \cdot 5 \cdot 5 \cdot 11}$$

$$a_3 = \frac{4a_2}{7 \cdot 17} = \frac{64a_0}{119 \cdot 825} = \frac{64a_0}{98175} = \frac{64a_0}{3 \cdot 5 \cdot 5 \cdot 11 \cdot 7 \cdot 17}$$

$$y_1 = a_0 x^{1/2}\left[1 + \frac{4}{15}x + \frac{16}{825}x^2 + \frac{64}{98175}x^3 + \ldots\right]$$

But, a_0 is arbitrary, so let's set $a_0 = 1$. This is the first of two solutions which will form a fundamental set of solutions.

$$y_1 = x^{1/2}\left[1 + \frac{4}{15}x + \frac{16}{825}x^2 + \frac{64}{98175}x^3 + \ldots\right]$$

$r_2 = 0$:
The recurrence relation becomes:

$$a_{n+1} = \frac{a_n}{(n+1)(3n+1)}, n = 0, 1, 2, 3, \ldots$$

$$a_0 = \text{arbitrary}$$

$$a_1 = \frac{a_0}{1}$$

$$a_2 = \frac{a_1}{2 \cdot 4} = \frac{a_0}{8} = \frac{a_0}{2 \cdot 4}$$

$$a_3 = \frac{a_2}{3 \cdot 7} = \frac{a_0}{8 \cdot 21} = \frac{a_0}{168} = \frac{a_0}{2 \cdot 3 \cdot 4 \cdot 7}$$

$$y_2 = a_0 x^0\left[1 + x + \frac{1}{8}x^2 + \frac{1}{168}x^3 + \ldots\right]$$

But, a_0 is arbitrary, so let's set $a_0 = 1$. This is the second of two solutions which will form a fundamental set of solutions.

$$y_2 = x^0\left[1 + x + \frac{1}{8}x^2 + \frac{1}{168}x^3 + \ldots\right]$$