e.g. Consider the equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (1+x)y = 0 \quad \text{about } 0$$

Let $y = c_0 x^r + \cdots + c_{n-1}x^{r+n-1} + c_n x^{r+n} + \cdots$

$$\frac{dy}{dx} = rc_0 x^{r-1} + \cdots + (r+n)c_n x^{r+n-1} + \cdots$$

$$\frac{d^2y}{dx^2} = r(r-1)c_0 x^{r-2} + \cdots + (r+n)(r+n-1)c_n x^{r+n-2} + \cdots$$

Then $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y + xy =$

$$[r(r-1)+3r+1]c_0x^r + \cdots + [\{(r+n)(r+n-1)+3(r+n)+1\}c_n$$
$$+ c_{n-1}]x^{r+n} + \cdots$$

The indicial equation is $r^2+2r+1=0$ so the index is -1

Then $\{(r+n)(r+n-1)+3(r+n)+1\}\ c_n + c_{n-1} = 0$

or $\{(n-1)(n-2)+3(n-1)+1\}c_n + c_{n-1} = 0$

so $c_n = -\frac{c_{n-1}}{n^2}$ for $n \geq 1$

Thus $c_1 = -c_0,\ c_2 = -\frac{c_1}{2^2} = \frac{c_0}{2^2},\ c_3 = -\frac{c_2}{3^2} = -\frac{c_0}{3^2 2^2}$

Therefore $y_1(x) = x^{-1}\left[1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \cdots\right]$

Then $[y_1(x)]^2 = x^{-2}\left[1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \cdots\right]\left[1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \cdots\right]$

$$= x^{-2}\left[1 + (-1-1)x + \left(\tfrac{1}{4}+1+\tfrac{1}{4}\right)x^2 + \left(-\tfrac{1}{36}-\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{36}\right)x^3 + \cdots\right]$$

$$= x^{-2}\left[1 - 2x + \tfrac{3}{2}x^2 - \tfrac{5}{9}x^3 + \cdots\right]$$

$$
\begin{array}{r|l}
 & 1 + 2x + \frac{5}{2}x^2 + \frac{23}{9}x^3 + \cdots \\
\hline
1 - 2x + \frac{3}{2}x^2 - \frac{5}{9}x^3 \cdots & 1 \\
 & 1 - 2x + \frac{3}{2}x^2 - \frac{5}{9}x^3 + \cdots \\
\hline
 & 2x - \frac{3}{2}x^2 + \frac{5}{9}x^3 + \cdots \\
 & 2x - 4x^2 + 3x^3 + \cdots \\
\hline
 & \frac{5}{2}x^2 - \frac{22}{9}x^3 + \cdots \\
 & \frac{5}{2}x^2 - 5x^3 + \cdots \\
\hline
 & \frac{23}{9}x^3 + \cdots
\end{array}
$$

Thus $\dfrac{1}{[y_1(x)]^2} = x^2\left[1 + 2x + \frac{5}{2}x^2 + \frac{23}{9}x^3 + \cdots\right]$

Then $e^{-\int \frac{a_1}{a_2}dx} = e^{-\int \frac{3x}{x^2}dx} = e^{-3\ln x} = \dfrac{1}{x^3}$

Therefore $\dfrac{e^{-\int \frac{a_1}{a_2}dx}}{[y_1(x)]^2} = \dfrac{1}{x}\left[1 + 2x + \frac{5}{2}x^2 + \frac{23}{9}x^3 + \cdots\right]$

$= \dfrac{1}{x} + 2 + \frac{5}{2}x + \frac{23}{9}x^2 + \cdots$

and $\displaystyle\int \frac{e^{-\int \frac{a_1}{a_2}dx}}{[y_1(x)]^2}dx = \ln x + 2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \cdots$

Finally $y_2(x) = y_1(x)\ln x + y_1(x)\left[2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \cdots\right]$

$= y_1(x)\ln x + x^{-1}\left[1 - x + \frac{x^2}{4} - \frac{x^3}{36} + \cdots\right]\left[2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \cdots\right]$

$= y_1(x)\ln x + x^{-1}\left[2x - \frac{3}{4}x^2 + \frac{11}{108}x^3 + \cdots\right]$ #

In general for this case ($r_1 = r_2$) the second solution $y_2(x)$ has the form

$$y_2(x) = y_1(x)\ln x + x^{r_1}\left(\sum_{n=1}^{\infty} b_n x^n\right)$$