This file contains example problems for each of the techniques we study in differential equations. Sometimes the method is discussed in general, and an example follows.

1 1^{st} Order: Separable Equations

The general first order equation is:

$$\frac{dy}{dx} = f(x, y),$$
$$M(x, y) + N(x, y)\frac{dy}{dx} = 0,$$

which is always possible by simply letting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, although other ways are possible.

Separable equations result if M is only a function of x, and N is only a function of y:

$$M(x) + N(y)\frac{dy}{dx} = 0,$$
$$M(x)dx + N(y)dy = 0,$$
$$M(x)dx = -N(y)dy,$$
$$\int M(x)dx = -\int N(y)dy,$$

and we can simply integrate both sides to get our answer.

Things to note: The distinction between dependent and independent variable is blurred. Also, we have replaced one integral with two which can (hopefully) perform.

1.1 Example

Solve

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}.$$

Notice that the $f(x, y) = g(x)/h(y)$; that is your clue to separability.

$$(1 - y^2)dy = x^2 dx,$$
$$\int (1 - y^2)dy = \int x^2 dx,$$
$$y - \frac{y^3}{3} + C_1 = \frac{x^3}{3} + C_2,$$
$$x^3 + y^3 - 3y + c = 0.$$

Is the implicit solution. Back substitute to check this result. Explicit solutions are sometimes impossible to find; as far as I am concerned, an implicit solution is as good as an explicit one, so don't bother trying to convert explicit case (you should, however, say that it is an implicit solution).

2 1$^{\text{st}}$ Order: Integrating Factor

Consider:

$$\frac{dy}{dt} + p(t)y = g(t).$$

Multiply by the integrating factor to get:

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t),$$

and note that the integrating factor satisfies:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + y\frac{d\mu(t)}{dt} \text{ which we want to } = \mu(t)g(t),$$

and comparing with what our original equation, we want to have:

$$\begin{aligned}
&\frac{d\mu(t)}{dt} = p(t)\mu(t), \\
&\frac{d\mu(t)}{\mu(t)} = p(t)\,dt, \\
&\int \frac{d\mu(t)}{\mu(t)} = \int p(t)\,dt, \\
&\ln|\mu(t)| = \int p(t)dt, \\
&\mu(t) = \exp(\int p(t)dt).
\end{aligned}$$

The integrating factor is written as the exponential of the integral of $p(t)$ (we have dropped the constants at this stage).

The final solution is found from solving:

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t),$$

$$y = \frac{\int \mu(s)g(s)ds|_{s=t} + c}{\mu(t)}.$$

We see that although for some cases the integrating factor method works well, it is certainly complicated, and may easily have integrals which can not be evaluated. In such cases we weren't able to find two simpler integrals to replace the one.

NOTE: Although we have just proved an interesting general result, you should not try and memorize it. It is easier to simply follow the procedure each time.

2.1 Example

Solve

$$\frac{dy}{dx} = \frac{1}{x + y^2}, \qquad y(-2) = 0.$$

Q: IS THIS LINEAR IN y? A: NO. Q: IS THIS LINEAR IN x? A: REWRITE IT, YES

$$\frac{dx}{dy} = x + y^2 \quad \text{or} \quad \frac{dx}{dy} - x = y^2$$

This is linear in x, so we will proceed with our integrating factor solution in terms of the dependent variable x.

$$\begin{aligned}
\mu(y)\frac{dx}{dy} - \mu(y)x = \mu(y)y^2, &\qquad \text{(multiply by integrating factor)}\\
\frac{d}{dy}[\mu(y)x] = \mu(y)\frac{dx}{dy} + \frac{d\mu}{dy}x, &\qquad \text{(chain rule–compare)}\\
\frac{d\mu}{dy} = -\mu, &\qquad \text{(DE for integrating factor)}\\
\mu(y) = \exp(-y) &\qquad \text{(integrating factor)}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dy}[\exp(-y)x] &= y^2 \exp(-y),\\
\exp(-y)x &= \int y^2 \exp(-y)dy,\\
&= -y^2 \exp(-y) - 2y\exp(-y) - 2\exp(-y) + c,\\
x &= -y^2 - 2y - 2 + c\exp(y),
\end{aligned}$$

and the initial condition is $x = -2, y = 0$ gives us $c = 0$, so we arrive at the implicit solution for y(x):

$$x = -y^2 - 2y - 2.$$

You should always back substitute and check your answer.

3 1^{st} Order: Exact Equations

This is a rather special trick, and won't work on most first order equations.

Theorem Let the functions M, N, M_y, N_x, where subscripts denote partial derivatives, be continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. Then

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is an exact differential equation in R if and only if

$$M_y(x,y) = N_x(x,y),$$

at each point of R. That is, there exists a function ψ satisfying

$$\psi_x(x,y) = M(x,y), \quad \psi_y(x,y) = N(x,y)$$

if and only if M and N satisfy

$$M_y(x,y) = N_x(x,y).$$

The solution of an exact differential equation is given implicitly by

$$\psi(x,y) = c.$$

The proof contains the method of solution of exact differential equations.

First, prove (2) implies (1). (2) tells us that:

$$M_y(x,y) = \psi_{xy}(x,y) = \psi_{yx}(x,y) = N_x(x,y)$$

So we have that direction easily.

Now prove that (1) implies the equations are exact. Assume we can construct the function ψ that satisfies (2):

$$\psi_x(x,y) = M(x,y), \quad \psi_y(x,y) = N(x,y)$$

We now integrate the first equation, holding y constant.

$$\psi(x,y) = \int M(x,y)dx + h(y)$$

The function $h(y)$ is an arbitrary function of y, and is really our constant of integration (y is held constant while we integrate with respect to x).

Now we show that we can always choose $h(y)$ so that $\psi_y(x,y) = N(x,y)$.

$$\begin{aligned}
\psi_y(x,y) &= \frac{\partial}{\partial y}\int M(x,y)dx + h'(y) \\
&= \int M_y(x,y)dx + h'(y) \\
N(x,y) &= \int M_y(x,y)dx + h'(y) \\
h'(y) &= N(x,y) - \int M_y(x,y)dx
\end{aligned}$$

Despite the appearance of what seems to be a function of x on the right hand side, the right hand side is actually only a function of y. You can prove this by taking the derivative with respect to x, and showing that it equals zero.

The function ψ is therefore given by

$$\psi(x,y) = \int M(x,y)dx + \int \left[N(x,y) - \int M_y(x,y)dx\right] dy.$$

3.1 Example

Solve the differential equation

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\frac{dy}{dx} = 0.$$

Identify M and N:

$$M(x,y) = y\cos x + 2xe^y, \quad N(x,y) = \sin x + x^2e^y - 1$$

$$M_y(x,y) = \cos x + 2xe^y = N_x(x,y)$$

So the DE is exact. Therefore, we look for a ψ that satisfies:

$$\psi_x(x,y) = M(x,y) = y\cos x + 2xe^y$$

$$\psi_y(x,y) = N(x,y) = \sin x + x^2e^y - 1$$

Let's integrate the second equation, to get:

$$\psi(x,y) = \int N(x,y)dy = y\sin x + x^2e^y - y + h(x)$$

Now, we differentiate with respect to x:

$$\psi_x(x,y) = M(x,y) = y\cos x + 2xe^y + \frac{dh(x)}{dx}$$

Now, we compare the two equations for $M(x,y)$, and identify that $h(x)$ must solve the equation:

$$\frac{dh(x)}{dx} = 0$$

$$h(x) = 0 + C = 0$$

We do not include any constants of integration here, much like when we dropped the constants of integration when we solved the differential equation for the integrating factor. The constants are all collected in the final solution

And so the complete solution is given implicitly by:

$$\psi(x,y) = C \rightarrow y\sin x + x^2e^y - y = C.$$

We made a choice to integrate the second equation, and you should verify that if we had instead integrated the first equation,

$$\psi_x(x,y) = M(x,y) = y\cos x + 2xe^y$$

we would still get the same result.

4 1^{st} Order: Exact Equations with Integrating Factors

We can use the technique of integrating factors to make an equation which is not exact, exact. This is a method that is extremely specialized, requires you to solve a partial differential equation, and so has limited utility. If the partial differential equation can not be solved by inspection (meaning, an educated guess will do) then trying the integrating factor technique on an exact equation simply restates the problem in terms of another differential equation you can't solve.

Let's take a look at the method.

If we have a differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

which is not exact, we can multiply it by an integrating factor $\mu(x,y)$:

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

and we try to choose the integrating factor so our new equation is exact. For this to be exact, we require:

$$(\mu M)_y = (\mu N)_x$$

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0$$

This is the partial differential equation that we must solve in order to determine the integrating factor $\mu(x, y)$. As you can see, it appears quite complicated!

We usually look for $\mu(x, y) = \mu(x)$ or $\mu(x, y) = \mu(y)$. The partial differential equation we must solve simplifies in these cases to an ODE:

$$N\mu_x + (M_y - N_x)\mu = 0,$$

or

$$M\mu_y + (M_y - N_x)\mu = 0,$$

which we can write as:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu,$$

or

$$\frac{d\mu}{dy} = \frac{M_y - N_x}{M}\mu.$$

Now, we can see that if

$$\frac{M_y - N_x}{N} = f(x)$$

we should look for an integrating factor which is only a function of x, since we can, in principle, solve the ODE for the integrating factor by using separation of variables.

Similarly, if

$$\frac{M_y - N_x}{M} = f(y)$$

we should look for an integrating factor which is only a function of y.

4.1 Example

Find an integrating factor for the differential equation:

$$(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$$

You can verify that this equation is not exact. Should we look for an integrating factor of the form $\mu(y)$ or $\mu(x)$?

Compute:

$$\begin{aligned} \frac{M_y - N_x}{M} &= \frac{(3x + 2y) - (2x + y)}{3xy + y^2} \\ &= = \frac{x + y}{3xy + y^2} \neq f(y) \end{aligned}$$

so we don't have a simple $\mu(x, y) = \mu(y)$ solution.

Compute:

$$\begin{aligned} \frac{M_y - N_x}{N} &= \frac{(3x + 2y) - (2x + y)}{x^2 + xy} \\ &= = \frac{x + y}{x^2 + xy} \\ &= \frac{1}{x} = f(x) \end{aligned}$$

so we have a simple $\mu(x, y) = \mu(x)$ solution for the integrating factor!

Now, the ordinary differential equation for the integrating factor becomes:

$$\frac{d\mu}{dx} = \frac{\mu}{x}$$

$$\mu = x$$

and the original differential equation can be written as:

$$(3x^2y + xy^2) + (x^3 + x^2y)\frac{dy}{dx} = 0$$

which is exact. Verify the implicit solution is given as:

$$x^3y + \frac{1}{2}x^2y^2 = c.$$

5 2^{nd} Order Homogeneous Constant Coefficient

Consider:

$$ay'' + by' + cy = 0.$$

Let's look for solutions of the form $y = e^{rt}$, where r is to be determined.

$$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt} \\ y'' &= r^2 e^{rt} \end{aligned}$$

Substitute into the original equation:

$$(ar^2 + br + c)e^{rt} = 0$$

and since $e^{rt} \neq 0$, we have

$$(ar^2 + br + c) = 0$$

which is called the **characteristic equation** for the differential equation.

What does it mean? It means that if r is a root of the characteristic equation, then $y = e^{rt}$ is a solution of the differential equation.

The characteristic equation is quadratic, so it will have two roots, and they may be:

- real and different
- complex conjugates
- real and equal (repeated roots)

5.1 Real and Distinct Roots of Characteristic Equation

Assume the roots are r_1, r_2 which are real, and $r_1 \neq r_2$. The two solutions are:

$$y_1(t) = e^{r_1 t} \qquad y_2(t) = e^{r_2 t}$$

and we can write a general solution as:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

5.2 Complex Conjugate Roots of Characteristic Equation

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad b^2 - 4ac < 0$$

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu$$

where

$$\lambda = \frac{-b}{2a}, \qquad \mu = \frac{\sqrt{4ac - b^2}}{2a}, \qquad i = \sqrt{-1}.$$

Note that λ and μ are real.

We can still use the solutions we found before:

$$y_1(t) = e^{r_1 t} = e^{(\lambda + i\mu)t}, \qquad y_2(t) = e^{r_2 t} = e^{(\lambda - i\mu)t}$$

Now we have to understand what a complex exponential means. For this, we turn to *Euler's Formula.* Euler's Formula provides a way of examining complex exponentials, and is based on the Taylor expansion of the exponential function for a real argument:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad -\infty < t < \infty$$

What if we simply substituted $t \to it$? Then we would have:

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!}, \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} t^{2n-1}}{(2n-1)!}, \\ &= \cos t + i \sin t \quad \text{Euler's Formula} \end{aligned}$$

Therefore, we can write:

$$e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$$

Note that if $\mu = 0$, we reduce to the real valued exponential:

$$e^{\lambda t} = e^{\lambda t}(\cos 0 + i \sin 0) = e^{\lambda t}$$

The complex exponential obeys all the same laws of exponents and derivative laws as does the real exponential function:

$$\frac{d}{dx}e^{rt} = re^{rt}, \qquad r \text{ Complex.}$$

$e^{-3+6i} = e^{-3}(\cos 6 + i \sin 6) \approx 0.047 - 0.014i.$

So, what we have now are solutions to the original differential equation for complex roots of the characteristic equation. But our solutions are also complex, which we could work with but prefer not to. Our original differential equation was not complex, so why should the solutions be?

How to construct real valued solutions from the complex ones Theorem 3.2.2 said that if y_1 and y_2 were solutions to the differential equation, then any linear combination of y_1 and y_2 is also a solution.

$$\begin{aligned}\bar{y}_3(t) &= y_1(t) + y_2(t) = e^{\lambda t}(\cos \mu t + i \sin \mu t) + e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2e^{\lambda t} \cos \mu t \\ \bar{y}_4(t) &= y_1(t) - y_2(t) = e^{\lambda t}(\cos \mu t + i \sin \mu t) - e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= 2ie^{\lambda t} \sin \mu t\end{aligned}$$

Since we are not concerned with any constant multiple of the solutions, we choose:

$$\begin{aligned}y_3(t) &= e^{\lambda t} \cos \mu t \\ y_4(t) &= e^{\lambda t} \sin \mu t\end{aligned}$$

as the fundamental set of solutions, since

$$W(y_3, y_4)(t) = \mu e^{2\lambda t},$$

which is nonzero as long as $\mu \neq 0$, which is true for complex roots of the characteristic equation.

NOTE: y_3 and y_4 are just the real and imaginary parts of the the original complex solutions y_1 and y_2. The real and imaginary parts of the complex solution must satisfy the original differential equation individually, so the use of them as a fundamental set of solutions should not be surprising.

The general solution of the differential equation:

$$ay'' + by' + cy = 0,$$

which has roots of the characteristic equation

$$ar^2 + br + c = 0, \qquad r = \lambda \pm \mu$$

is

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t,$$

where c_1 and c_2 are arbitrary constants.

5.3 Real Root of Multiplicity Two of Characteristic Equation

We will still have one solution to the differential equation, given by $y_1(t) = e^{rt}$ where r is the root of the characteristic equation. We get a second solution using the method of reduction of order, which is outlined below.

Construct a second function

$$y = v(t) y_1(t) = v(t) e^{-bt/2a}$$

and our task is to determine $v(t)$ such that this is a solution of the original differential equation.

If it is to solve the differential equation, we should begin by substituting back into the differential equation:

$$\begin{aligned} y &= v(t)e^{-bt/2a} \\ y' &= v'(t)e^{-bt/2a} - \frac{b}{2a} v(t) e^{-bt/2a} \\ y'' &= v''(t)e^{-bt/2a} - \frac{b}{a} v'(t) e^{-bt/2a} + \frac{b^2}{4a^2} v(t) e^{-bt/2a} \end{aligned}$$

Substitute back into the differential equation to obtain:

$$\left[a \left(v''(t) - \frac{b}{a} v'(t) + \frac{b^2}{4a^2} v(t) \right) + b \left(v'(t) - \frac{b}{2a} v(t) \right) + cv(t) \right] e^{-bt/2a} = 0$$

The exponential is not zero, so we can cancel it. Rearranging yields:

$$av''(t) + (-b + b)v'(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c \right) v(t)$$

And

$$\begin{aligned} -b + b &= 0 \\ \frac{b^2}{4a} - \frac{b^2}{2a} + c &= -\frac{b^2}{4a} + c \\ &= \frac{-1}{4a} \left(b^2 - 4ac \right) = 0 \end{aligned}$$

So to satisfy the differential equation we require that

$$v''(t) = 0.$$

This has solution $v(t) = c_1 t + c_2$, and so

$$y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}.$$

The two solutions are $y_1(t) = e^{-bt/2a}$ and $y_2 = te^{-bt/2a}$. The Wronskian is $W(y_1, y_2)(t) = e^{-bt/2a}$, which is never zero, and so these two solutions form a fundamental set of solutions. The linear combination above is therefore the general solution to the differential equation.

5.4 Example

Find the solution of the initial value problem:

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = 3$$

Assume a solution of the form y_e^{rt}, then r must be a root of the characteristic equation:

$$r^2 + 5r + 6 = (r+2)(r+3) = 0 \longrightarrow r_1 = -2, r_2 = -3$$

so the general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

Apply the initial conditions:

$$\begin{array}{rcl} y(0) = 2 & \rightarrow & c_1 + c_2 = 2 \\ y'(0) = 3 & \rightarrow & -2c_1 - 3c_2 = 3 \end{array}$$

which has the solution: $c_1 = 9, c_2 = -7$. The solution to the initial value problem is therefore:

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

5.5 Example

Find the general solution of

$$y'' + y' + y = 0.$$

The characteristic equation is

$$r^2 + r + 1 = 0$$

which has roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} = \lambda \pm i\mu, \quad \lambda = -\frac{1}{2}, \mu = \frac{\sqrt{3}}{2}$$

The fundamental set of solutions is given by:

$$\begin{aligned} y_1(t) &= e^{\lambda t}\cos\mu t = e^{-t/2}\cos(\sqrt{3}t/2), \\ y_2(t) &= e^{\lambda t}\sin\mu t = e^{-t/2}\sin(\sqrt{3}t/2). \end{aligned}$$

The general solution is therefore given by:

$$y = c_1 e^{-t/2}\cos(\sqrt{3}t/2) + c_2 e^{-t/2}\sin(\sqrt{3}t/2)$$

6 n$^{\text{th}}$ Order Homogeneous Constant Coefficient

Consider the nth order linear homogeneous differential equation:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

where a's are constants. We expect that $y = e^{rt}$ is a solution. Substitute and we get the characteristic equation:

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n = 0$$

For r which satisfy the characteristic equation, we have e^{rt} as a solution.

An nth degree polynomial has n zeros, $r_1, r_2, \ldots, r_n$.

$$Z(r) = (r - r_1)(r - r_2)\cdots(r - r_n)$$

Some of the r_i may be equal.

6.1 Real and Unequal Roots

If roots are real, and no two are equal, then we have n distinct solutions. If these functions are linearly independent, then the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$$

We can check linear independence of the fundamental set of solutions using the Wronskian.

6.2 Complex Roots

If the characteristic equation has complex roots, they must occur in conjugate pairs $\lambda \pm i\mu$. We get complex roots appearing in complex conjugate pairs since the coefficients of the characteristic equation are real. If they are complex, complex roots need not occur in complex conjugate pairs. The real valued solutions for these complex roots are

$$e^{\lambda t}\cos\mu t, \qquad e^{\lambda t}\sin\mu t$$

The general solution is made up of a linear combination of these functions.

6.3 Repeated Roots

If a root r_1 has multiplicity s, then the s linearly independent functions that are solutions are

$$e^{r_1 t},\ te^{r_1 t},\ t^2 e^{r_1 t},\ \ldots,\ t^{s-1}e^{r_1 t}$$

This works for real or imaginary r_1.

6.4 Example

Solve the differential equation $y^{(4)} - y = 0$.

Assume $y = e^{rt}$.
Substitute into the differential equation: $r^4 e^{rt} - e^{rt} = 0$.
Characteristic equation: $r^4 - 1 = 0$.
Difference of squares: $(r^2-1)(r^2+1) = 0 \longrightarrow r_1 = i, r_2 = -i, r_3 = 1, r_4 = -1$.
r_1 and r_2 are complex conjugates, with $\lambda = 0, \mu = 1$.
The fundamental set of solutions is:
$y_1(t) = e^{\lambda t}\cos\mu t = \cos t$, $y_2(t) = e^{\lambda t}\sin\mu t = \sin t$, $y_3(t) = e^{r_3 t} = e^t$, $y_4(t) = e^{r_4 t} = e^{-t}$.
The general solution is $y(t) = c_1\cos t + c_2\sin t + c_3 e^t + c_4 e^{-t}$.

6.5 Example

Solve the differential equation $y^{(4)} + 2y'' + y = 0$.

Assume $y = e^{rt}$.
Substitute into the differential equation: $r^4 e^{rt} + 2r^2 e^{rt} + e^{rt} = 0$.
Characteristic equation: $r^4 + 2r^2 + 1 = 0$.
(Quadratic in r^2): $(r^2+1)(r^2+1) = (r^2+1)^2 = 0$.
$(r^2+1) = 0$ of multiplicity 2.
$r_1 = +i$, $r_2 = -i$, of multiplicity 2.

The roots are $r_1 = i, r_2 = -i, r_3 = i, r_4 = -i$.
r_1 and r_2 are complex conjugates, with $\lambda = 0, \mu = 1$.
The fundamental set of solutions is:
$y_1(t) = e^{\lambda t}\cos\mu t = \cos t,\ y_2(t) = e^{\lambda t}\sin\mu t = \sin t,\ y_3(t) = t\cos t,\ y_4(t) = t\sin t.$
The general solution is $y(t) = c_1\cos t + c_2\sin t + c_3 t\cos t + c_4 t\sin t$.

7 Reduction of Order

The method of reduction of order will work on n^{th} order differential equations, however, it is mainly used on second order equations.

Suppose we have the DE

$$y'' + p(t)y' + q(t)y = 0$$

where p, q are continuous on some interval I. Let us assume that $y_1(t)$ is a known solution of the differential equation, and that $y_1(t) \neq 0$ for all t in I. We construct a solution of the form:

$$\begin{aligned} y &= u(t)y_1(t) \\ y' &= uy'_1 + y_1u' \\ y'' &= uy''_1 + 2y'_1u' + y_1u'' \end{aligned}$$

where our job is to determine u such that y satisfies the differential equation. Substituting into the differential equation, we obtain:

$$\begin{aligned} y'' + py' + qy &= u[y''_1 + py'_1 + qy_1] + y_1u'' + (2y'_1 + py_1)u' = 0 \\ 0 &= y_1u'' + (2y'_1 + py_1)u' \end{aligned}$$

We make the substitution $w = u'$. This is the reduction of order part. The new equation in w is linear and separable:

$$\begin{aligned} & y_1w' + (2y'_1 + py_1)w = 0 \\ & y_1\frac{dw}{dt} + (2y'_1 + py_1)w = 0 \\ & \frac{dw}{w} = -(2\frac{y'_1}{y_1} + p)dt \\ & \int\frac{dw}{w} = -\int(2\frac{y'_1}{y_1} + p)dt \\ & \ln|w| = -2\int\frac{dy_1}{y_1} - \int p\,dt + C \\ & \ln|w| = -2\ln|y_1| - \int p\,dt + C \\ & \ln|wy_1^2| = -\int p\,dt + C \end{aligned}$$

$$wy_1^2 = c_1 \exp\left(-\int p\,dt\right)$$
$$w = c_1 \frac{\exp\left(-\int p\,dt\right)}{y_1^2}$$

We still need to integrate once more to get the u, since $u' = w$:

$$u = c_1 \int \frac{\exp\left(-\int p\,dt\right)}{y_1^2} dt + c_2$$

The solution to the differential equation is given by:

$$\begin{aligned} y &= u(t)y_1(t) \\ &= c_1 y_1(t) \int \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)} dt + c_2 y_1(t) \end{aligned}$$

Now, two solutions of the differential equation are

$$y_1(t) \text{ (given or known) }, \qquad y_2(t) = y_1(t) \int \frac{\exp\left(-\int p(t)\,dt\right)}{y_1^2(t)} dt$$

8 Nonhomogeneous Differential Equations

The Process to solve second order nonhomogeneous equations:

1. Find the general solution of the corresponding homogeneous equation. This solution is called the *complementary* solution, and is denoted $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$.
2. Find some singular solution $Y(t)$ of the nonhomogeneous equation. This is called the *particular* solution, and is denoted $y_p(t) = Y(t)$.
3. The general solution to the nonhomogeneous equation is given by $y(t) = y_c(t) + y_p(t)$.

9 Undetermined Coefficients

Summary of how to use undetermined coefficients to find a particular solution:

- If $g(t) = e^{\beta t}$, assume the particular solution is proportional to $e^{\beta t}$.
- If $g(t) = \sin \beta t, \cos \beta t$, assume the particular solution is proportional to a linear combination of $\sin \beta t$ and $\cos \beta t$.
- If $g(t)$ is a polynomial, than assume the particular solution is a polynomial of like degree.

- If $g(t)$ is a product of the above forms, assume the particular solution is the corresponding product.
- If $g(t)$ has more than one term, split the differential equation up and solve for a particular solution for each term individually.
- If the assumed solution has any part which appears in the complementary solution, the method will fail; multiply the assumed solution by t^n until there is no overlap with the complementary solution.

The method is essentially the same for higher order differential equations.

9.1 Example

Find the general solution of the differential equation $u'' + w_0^2 u = \cos(wt)$, where $w_0^2 \neq w^2$.

First, solve the associated homogeneous equation $u'' + w_0^2 u = 0$.
Assume $u = e^{rt}$.
Substitute into the differential equation: $r^2 e^{rt} + w_0^2 e^{rt} = 0$.
Characteristic equation: $r^2 + w_0^2 = 0$.
Roots of the characteristic equation are complex: $r = \pm w_0 i = \lambda \pm \mu i$.
Therefore, $\lambda = 0, \mu = w_0$.
A fundamental set of solutions is $u_1 = \cos w_0 t$, $u_2 = \sin w_0 t$.
The complementary solution is therefore $u_c(t) = c_1 \cos w_0 t + c_2 \sin w_0 t$.

Get a particular solution of the nonhomogeneous equation. Assume $U(t) = A \cos wt + B \sin wt$.
Substitute into the differential equation:

$$\begin{aligned}
(-Aw^2 \cos wt - Bw^2 \sin wt) + w_0^2(A \cos wt + B \sin wt) &= \cos wt \\
A(w_0^2 - w^2) \cos wt + B(w_0^2 - w^2) \sin wt &= \cos wt \\
\rightarrow \quad A(w_0^2 - w^2) = 1 & \qquad B(w_0^2 - w^2) = 0 \\
A = \frac{1}{w_0^2 - w^2} & \qquad B = 0
\end{aligned}$$

The particular solution of the nonhomogeneous differential equation is

$$y_p(t) = \frac{1}{w_0^2 - w^2} \cos wt$$

and the general solution is

$$y(t) = y_c(t) + y_p(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + \frac{1}{w_0^2 - w^2} \cos wt.$$

10 Variation of Parameters on 1st Order Equations

To begin, let's look at the method applied to something we have already seen, namely the first order equation:

$$y' + p(t)y = g(t)$$

where p, g are continuous on I.

The equation with $f(t) = 0$

$$y' + p(t)y = 0$$

has the solution

$$y_c(t) = c_1 \exp(-\int p(t)dt) = c_1 y_1(t).$$

We want to write the solution of the equation where $f(t) \neq 0$ as

$$y(t) = y_c(t) + y_p(t)$$

Variation of parameters consists of finding a particular solution such that:

$$y_p(t) = Y(t) = u_1(t)y_1(t).$$

What we have done is changed the constant in the complementary solution to a variable function of t.

Differentiate and substitute into the nonhomogeneous differential equation:

$$\begin{aligned} Y(t) &= u_1(t)y_1(t) \\ Y'(t) &= u_1'(t)y_1(t) + u_1(t)y_1'(t) \end{aligned}$$

$$u_1(t)[y_1' + p(t)y_1] + y_1 u_1' = g(t)$$

Since $y_1(t)$ solves the homogeneous equation, the first term is zero, and we have:

$$y_1 u_1' = g(t)$$

We can immediately separate variables and integrate to determine:

$$u_1(t) = \int \frac{g(t)}{y_1(t)} dt + c_1$$

Setting the constant of integration $c_1 = 0$ (since we are looking for a particular solution), we arrive at:

$$y_p(t) = u_1(t)y_1(t) = y_1(t) \int \frac{g(t)}{y_1(t)} dt$$

Collecting everything together, we see that the general solution to the nonhomegeneous equation is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= c_1 y_1(t) + y_1(t) \int \frac{g(t)}{y_1(t)} dt \end{aligned}$$

Look back at the integration factor technique we used (pg 2). Notice the relation between $y_1(t)$ and $\mu(t)$: $y_1(t) = \mu(t)^{-1}$. *Variation of parameters on a first order differential equation is just the integrating factor technique!*

11 Variation of Parameters on 2$^{\text{nd}}$ Order Equations

As we might guess, implementation requires the integration of the nonhomogeneous term which may prove impossible to do.

The Method Consider

$$y'' + p(t)y' + q(t)y = g(t),$$

where p, q, g are continuous on I. Assume that we know the complementary solution of the associated homogeneous equation ($g(t) = 0$):

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t).$$

The basic idea is to replace in the complementary solution the constants c_1, c_2 with the functions $u_1(t), u_2(t)$:

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

and then we try and determine u_1, u_2 so that this solves the nonhomogeneous equation (this is then a particular solution).

To determine u_1, u_2, we need to differentiate and substitute back into the differential equation. This will give us a single differential equation involving u_1, u_2 and their first two derivatives. This is one differential equation and two unknowns, so it will be underdetermined, and there will exist more than one set of functions $u_1(t), u_2(t)$ that will solve the equation.

We can therefore introduce a second condition of our own choosing, to result in two equations for two unknowns, which will have a unique solution u_1, u_2.

Differentiate:

$$Y'(t) = u_1'(t)y_1(t) + u_1(t){y'}_1(t) + u_2'(t)y_2(t) + u_2(t){y'}_2(t)$$

Introduce our own second condition:

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

which simplifies the differential equation to:

$$Y'(t) = u_1(t){y'}_1(t) + u_2(t){y'}_2(t)$$

Differentiate a second time:

$$Y''(t) = u_1'(t){y'}_1(t) + u_1(t){y''}_1(t) + u_2'(t){y'}_2(t) + u_2(t){y''}_2(t)$$

Substitute into the differential equation, and we get:

$$\begin{aligned} &u_1(t)[{y''}_1(t) + p(y){y'}_1(t) + q(t)y_1(t)] + \\ &u_2(t)[{y''}_2(t) + p(y){y'}_2(t) + q(t)y_2(t)] + \\ &u_1'(t){y'}_1(t) + u_2'(t){y'}_2(t) = g(t) \end{aligned}$$

The first two terms are zero since y_1, y_2 solve the homogeneous equation. We are left with:

$$u_1'(t){y'}_1(t) + u_2'(t){y'}_2(t) = g(t).$$

We now have:

$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) &= g(t) \end{aligned}$$

This is an algebraic system of two equation for the two unknowns $u_1'(t), u_2'(t)$. We made the particular choice for the second condition that we did so as to arrive at this set of equations. The solution is given by:

$$u_1'(t) = \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)}, \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$

where W is the Wronskian, which is nonzero since y_1, y_2 form a fundamental set of solutions. The functions u_1, u_2 are found by integrating to be:

$$u_1(t) = \int \frac{-y_2(t)g(t)}{W(y_1, y_2)(t)} dt + c_3 \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt + c_4$$

We can set the constants of integration c_3, c_4 equal to zero since we are solving for the particular solution. Substituting back into the original expression, we find the particular solution to be:

$$y_p(t) = Y(t) = -y_1(t) \int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt + y_2(t) \int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt$$

11.1 Example

Find the general solution for

$$y'' - 4y' + 4y = (t+1)e^{2t},$$

First, solve the associated homogeneous equation $y'' - 4y' + 4y = 0$.
Assume $y = e^{rt}$.
Substitute into the differential equation: $r^2e^{rt} - 4re^{rt} + 4e^{rt} = 0$.
Characteristic equation: $r^2 + -4r + 4 = (r-2)^2 = 0$.
Root of the characteristic equation is: $r = 2$ of multiplicity 2.
A fundamental set of solutions is $y_1 = e^{2t}$, $y_2 = te^{2t}$.
The complementary solution is therefore $y_c(t) = c_1e^{2t} + c_2te^{2t}$.

Variation of parameters: replace in the complementary solution the constants c_1, c_2 with the functions $u_1(t), u_2(t)$:

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1e^{2t} + u_2te^{2t},$$

and then we try and determine u_1, u_2 so that this solves the nonhomogeneous equation (this is then a particular solution).

Differentiate:

$$Y'(t) = e^{2t}(2u_1 + u_2 + 2tu_2 + u_1' + tu_2')$$

Introduce our own second condition:

$$u_1' + tu_2' = 0$$

which simplifies the differential equation to:

$$Y'(t) = e^{2t}(2u_1 + u_2 + 2tu_2)$$

Differentiate a second time:

$$Y''(t) = e^{2t}(4u_1 + 4(1+t)u_2 + 2u_1' + u_2' + 2tu_2')$$

Substitute into the differential equation, and we get upon simplification:

$$u_1'(2e^{2t}) + u_2'(e^{2t} + 2te^{2t}) = (t+1)e^{2t}$$

We now have a system of equations for u_1' and u_2':

$$\begin{aligned} u_1'(e^{2t}) + \quad & u_2'(te^{2t}) = 0 \\ u_1'(2e^{2t}) + \quad & u_2'(e^{2t} + 2te^{2t}) = (t+1)e^{2t} \end{aligned}$$

$$u_1' = \frac{\begin{vmatrix} 0 & te^{2t} \\ (t+1)e^{2t} & e^{2t}+2te^{2t} \end{vmatrix}}{\begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}+2te^{2t} \end{vmatrix}} = -(t^2+t)$$

$$u_2' = \frac{\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & (t+1)e^{2t} \end{vmatrix}}{\begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}+2te^{2t} \end{vmatrix}} = t+1$$

Integrate to find:

$$u_1 = -\int (t^2+t)dt = -\frac{t^3}{3} - \frac{t^2}{2}$$
$$u_2 = \int (t+1)dt = \frac{t^2}{2} + t$$

The particular solution is therefore given by:

$$y_p(t) = u_1y_1 + u_2y_2 = -(\frac{t^3}{3} + \frac{t^2}{2})e^{2t} + (\frac{t^2}{2} + t)te^{2t} = (\frac{t^3}{6} + \frac{t^2}{2})e^{2t}$$

The general solution is given by:

$$y(t) = y_c(t) + y_p(t) = c_1e^{2t} + c_2te^{2t} + (\frac{t^3}{6} + \frac{t^2}{2})e^{2t}.$$

12 Variation of Parameters on $\mathbf{n^{th}}$ Order Equations

Assume we know a fundamental set of solutions $y_1, y_2, \ldots, y_n$ for the homogeneous equation. Then the general complementary solution is

$$y_c(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t).$$

Assume a particular solution of the nonhomogeneous equation exists of the form:

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t).$$

Since we have n functions u_n to determine, we shall have to specify n conditions. One condition is that $Y(t)$ satisfy the differential equation:

$$L[Y] = g(t).$$

The other $n-1$ conditions are chosen to simplify the calculation as much as possible.

Everything is a function of time, so drop that notation:

$$Y = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

Now start taking derivatives:

- First Condition: $u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0$
- First Derivative: $Y' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'$
- Second Condition: $u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0$
- Second Derivative: $Y^{(2)} = u_1 y_1^{(2)} + u_2 y_2^{(2)} + \cdots + u_n y_n^{(2)}$

Continue this procedure, to get the following:

- $n-1$ conditions: $u_1' y_1^{(m)} + u_2' y_2^{(m)} + \cdots + u_n' y_n^{(m)} = 0, \quad m = 0, 1, 2, \ldots, n-2$
- $n-1$ derivatives: $Y^{(m)} = u_1 y_1^{(m)} + u_2 y_2^{(m)} + \cdots + u_n y_n^{(m)}, \quad m = 0, 1, 2, \ldots, n-1$

The n^{th} derivative is therefore:

$$Y^{(n)} = (u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)}) + (u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)})$$

Substitute all this into the differential equation, collect terms, use $L[y_i] = 0$, and you will arrive at

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g$$

This equation plus the $n-1$ conditions gives an algebraic system, n equations for the n unknowns u_i':

$$\begin{aligned}
&u_1' y_1^{(0)} + u_2' y_2^{(0)} + \cdots + u_n' y_n^{(0)} = 0 \\
&u_1' y_1^{(1)} + u_2' y_2^{(1)} + \cdots + u_n' y_n^{(1)} = 0 \\
&u_1' y_1^{(2)} + u_2' y_2^{(2)} + \cdots + u_n' y_n^{(2)} = 0 \\
&\vdots \\
&u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)} = 0 \\
&u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g
\end{aligned}$$

The existence of a solution of this algebraic system is that $W(y_1, y_2, \ldots, y_n) \neq 0$, which have since the y_i form a fundamental set of solutions (linearly independent).

The solution to the system is found using **Cramer's Rule**:

$$u'_m(t) = \frac{g(t)W_m(t)}{W(t)}, m = 1, 2, \ldots, n,$$

where $W(t) = W(y_1, y_2, \ldots, y_n)$ and $W_m(t)$ is found from $W(t)$ by replacing the mth column by the column (0,0,0,...,1). This is because the right hand side of the system of equations are all zero except for the one that is $g(t)$, and we have factored out the $g(t)$ in the equation for $u'_m(t)$.

A particular solution of the nonhomogeneous equation is given by

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)} ds.$$

12.1 Example

Find the solution to $y''' - y' = t$ by variation of parameters.

First, solve the homogeneous equation for the complementary solution.
Assume $y(t) = e^{rt}$.
Characteristic equation: $r^3 - r = r(r-1)(r+1) = 0$.
Roots are $r_1 = 0, r_2 = +1, r_2 = -1$.
A fundamental set of solutions is $y_1 = 1, y_2 = e^t, y_3 = e^{-t}$.
The complementary solution is therefore: $y_c(t) = c_1 + c_2e^t + c^3e^{-t}$.

The Wronskain of the fundamental set of solutions is:

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2$$

Get a particular solution of the form $Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$.
Note that the nonhomogeneous term is $g(t) = t$.

<u>$Y_1(t)$:</u>

$$W_1(y_1, y_2, y_3)(t) = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2$$

$$\int_{t_0}^{t} \frac{g(s)W_1(s)}{W(s)} ds = \int_{t_0}^{t} \frac{s(-2)}{2} ds = -\int_{t_0}^{t} s ds = -\frac{t^2}{2} - \frac{t_0^2}{2}$$

Since t_0 is arbitrary, let's set it equal to zero, and we have:

$$Y_1(t) = y_1 \int_{t_0}^{t} \frac{g(s)W_1(s)}{W(s)} ds = y_1(t)\left(-\frac{t^2}{2}\right) = -\frac{t^2}{2}$$

$\underline{Y_2(t)}$:

$$W_2(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t}$$

$$\int_{t_0}^{t} \frac{g(s)W_2(s)}{W(s)} ds = \int_{0}^{t} \frac{se^{-s}}{2} ds = \frac{1}{2}(1 - e^{-t}(1+t))$$

$$Y_2(t) = y_2 \int_{t_0}^{t} \frac{g(s)W_2(s)}{W(s)} ds = \frac{e^t}{2} - \frac{(1+t)}{2}$$

$\underline{Y_3(t)}$:

$$W_3(y_1, y_2, y_3)(t) = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = e^t$$

$$\int_{t_0}^{t} \frac{g(s)W_3(s)}{W(s)} ds = \int_{0}^{t} \frac{se^{s}}{2} ds = \frac{1}{2}(1 + e^{t}(t-1))$$

$$Y_3(t) = y_3 \int_{t_0}^{t} \frac{g(s)W_3(s)}{W(s)} ds = \frac{e^{-t}}{2} + \frac{(t-1)}{2}$$

The complete particular solution is given by

$$Y(t) = Y_1(t) + Y_2(t) + Y_3(t) = -\frac{t^2}{2} + \frac{e^t}{2} + \frac{e^{-t}}{2} - 1$$

The last three terms are part of the complementary solution and so can be dropped from the particular solution. The particular solution we choose will be $Y(t) = -t^2/2$.

The general solution is

$$y(t) = y_c(t) + y_p(t) = c_1 + c_2 e^t + c_3 e^{-t} - \frac{t^2}{2}$$

Note: This example is much simpler using undetermined coefficients:

Assume a particular solution looks like: $Y(t) = At^2 + Bt$.
$Y'(t) = 2At + B$.
$Y''(t) = 2A$.
$Y'''(t) = 0$.
Substitute into the differential equation: $y''' - y' = 0 - 2At - B = t$.
Therefore, $B = 0, A = -1/2$.
A particular solution is therefore $y_p(t) = -t^2/2$.

13 Ordinary and Regular Singular Points

These classifications are for second order differential equations.

Consider the differential equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

The point x_0 is an ordinary point if

$$p(x) = \frac{Q(x)}{P(x)} \text{ and } q(x) = \frac{R(x)}{P(x)}$$

are analytic about $x = x_0$ (convergent Taylor series in some nonzero interval). If $x = x_0$ is not ordinary, it is a singular point. We say x_0 is a regular singular point if

$$(x - x_0)p(x) \text{ and } (x - x_0)^2 q(x)$$

are analytic about $x = x_0$.

Irregular singular points are singular points which are not regular. There is no comprehensive theory for irregular singular points (although you could try techniques like asymptotics and dominant balance).

You can also classify the point $x_0 = \infty$ by making the transformation $x = 1/t$. This is called finding singularities at infinity.

If you are interested in these methods, take a look at Advanced Mathematical Methods for Scientists and Engineers (Asymptotic Methods and Perturbation Theory), Bender and Orszag. It was reprinted in 1999, and I think it is a wonderful book.

13.1 Example

Consider the differential equation:

$$(x + 2)^2(x - 1)y'' + 3(x - 1)y' - 2(x + 2)y = 0.$$

Find and classify all the singular points.

Solution:

$$p(x) = \frac{Q(x)}{P(x)} = \frac{3(x - 1)}{(x + 2)^2(x - 1)} = \frac{3}{(x + 2)^2}$$

has a singular point at $x_0 = -2$.

$$q(x) = \frac{R(x)}{P(x)} = \frac{-2(x + 2)}{(x + 2)^2(x - 1)} = \frac{-2}{(x + 2)(x - 1)}$$

has singular points at $x_0 = -2, +1$.

The singular points are $x_0 = -2, +1$.

Classify: $\underline{x_0 = -2}$:

$$(x - x_0)p(x) = (x+2)\frac{3}{(x+2)^2} = \frac{3}{x+2}$$

$$(x - x_0)^2 q(x) = (x+2)^2 \frac{-2}{(x+2)(x-1)} = \frac{-2(x+2)}{x-1}$$

The second has a convergent Taylor series about $x_0 = -2$, but the first does not. Therefore $x_0 = -2$ is an irregular singular point.

Classify: $\underline{x_0 = +1}$:

$$(x - x_0)p(x) = (x-1)\frac{3}{(x+2)^2} = \frac{3(x-1)}{(x+2)^2}$$

$$(x - x_0)^2 q(x) = (x-1)^2 \frac{-2}{(x+2)(x-1)} = \frac{-2(x-1)}{(x+2)}$$

These both have convergent Taylor series for some nonzero interval about $x_0 = +1$, so $x_0 = +1$ is a regular singular point.

14 Euler Equations

Consider the differential equation $L[y] = x^2 y'' + \alpha x y' + \beta y = 0$ where α, β are real constants. The point $x_0 = 0$ is a regular singular point since

$$p(x) = \frac{\alpha}{x}, \qquad q(x) = \frac{\beta}{x^2}$$

are not analytic at $x = 0$, so $x = 0$ is a singular point.

$$xp(x) = x\frac{\alpha x}{x^2} = \alpha, \qquad x^2 q(x) = x^2 \frac{\beta}{x^2} = \beta$$

are both analytic about $x_0 = 0$, so $x = 0$ is a regular singular point.

In any interval not containing the origin, the solution of Euler's equation is

$$y = c_1 y_1 + c_2 y_2$$

where y_1, y_2 are linearly independent. For now, let's take $x > 0$. Note that

$$y = x^r, \qquad y' = rx^{r-1}, \qquad y'' = r(r-1)x^{r-2}$$

So let's assume that the equation has a solution of the form x^r.

$$L[x^r] = x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = x^r[r(r-1) + \alpha r + \beta]$$

If r is a root of the quadratic equation

$$F(r) = r(r-1) + \alpha r + \beta = 0$$

then x^r is a solution of Euler's equation. The roots are

$$r_{1,2} = \frac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

You should see that this is the same method we used when we assumed a solution of the form e^t for second order equations and derived a characteristic equation. This equation we seek roots of is similar to the characteristic equation, and we call it the *indicial equation*.

We need to look at the cases: real distinct roots; real repeated roots; complex roots.

14.1 Real Distinct Roots

If $r_1 \neq r_2$ and both are real, then $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$ are two solutions of Euler's equation. Their Wronskian is nonzero, so they are linearly independent, and we can write the general solution as:

$$y = c_1 x^{r_1} + c_2 x^{r_2}, \qquad x > 0$$

14.2 Equal roots

If the two roots of the indicial equation are equal, we can only write one solution to Euler's equation,

$$y_1 = x^{r_1}, \qquad x > 0$$

The Method of Reduction of Order would allow us to determine a second solution:

$$y_2 = x^{r_1} \ln x, \qquad x > 0$$

The Wronskian of the two solutions is nonzero, so they are linearly independent. The general solution of the differential equation is therefore:

$$y = (c_1 + c_2 \ln x) x^{r_1}.$$

14.3 Complex Roots

Complex roots occur in complex conjugate pairs for the Euler equation, since the coefficients of the equation are real,

$$r_{1,2} = \lambda \pm \mu i.$$

We can write the solution as

$$y = c_1 x^{\lambda+\mu i} + c_2 x^{\lambda-\mu i}$$

This is a complex valued function, and we would prefer to have real valued functions since the Euler equation is a real valued differential equation.

$$x^{\lambda+\mu i} = x^\lambda x^{\mu i} = x^\lambda e^{\mu i \ln x} = x^\lambda[\cos(\mu \ln x) + i \sin(\mu \ln x)]$$

The real and imaginary parts are both solutions, so we can instead use the real valued solutions

$$y_1 = x^\lambda \cos(\mu \ln x), \qquad y_2 = x^\lambda \sin(\mu \ln x)$$

which we can show are linearly independent, and so the general solution is

$$y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x)$$

Extension to $x < 0$ The solutions immediately transfer to the region $x < 0$, but could become complex valued (think of $x^{1/2}$). The change of variable $x = -\xi, \xi > 0$, and we obtain real valued solutions.

The general solution of the Euler equation

$$x^2 y'' + \alpha x y' + \beta y = 0$$

in any interval not containing the origin is determined by the roots r_1 and r_2 of the equation

$$F(r) = r(r-1) + \alpha r + \beta = 0.$$

If the roots are real and distinct, the general solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$.

If the roots are real and equal, the general solution is $y = (c_1 + c_2 \ln|x|)|x|^{r_1}$.

If the roots are complex, the general solution is $y = |x|^\lambda[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)]$, where $r_{1,2} = \lambda \pm \mu i$.

14.4 Example

Solve the differential equation:

$$x^2y'' + 6xy' - y = 0, \qquad x > 0.$$

This is an Euler equation.
Assume the solution looks like $y = x^r$.
Differentiate and substitute into the differential equation.
$r(r-1)x^r + 6rx^r - x^r = 0$.
Indicial equation: $r^2 - r + 6r - 1 = r^2 + 5r - 1 = 0$.
The roots of the indicial equation are:
$r_1 = \frac{1}{2}(-5 - \sqrt{29}) =$ -5.19258,
$r_2 = \frac{1}{2}(-5 + \sqrt{29}) = 0.19258$,
two real, distinct roots.
The general solution is therefore:
$y = c_1x^{r_1} + c_2x^{r_2} = c_1x^{-5.19258} + c_2x^{0.19258}$.

14.5 Example

Solve the differential equation:

$$x^2y'' + 3xy' + 5y = 0.$$

This is an Euler equation.
Assume the solution looks like $y = x^r$.
Differentiate and substitute into the differential equation.
$r(r-1)x^r + 3rx^r + 5x^r = 0$.
Indicial equation: $r^2 - r + 3r + 5 = r^2 + 2r + 5 = 0$.
The roots of the indicial equation are:
$r_1 = -1 - 2i$,
$r_2 = -1 + 2i$,
two complex conjugate roots. We have $\lambda = -1, \mu = 2$.
The general solution is therefore:
$y = |x|^\lambda[c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)] = |x|^{-1}[c_1\cos(2\ln|x|) + c_2\sin(2\ln|x|)]$.

15 Series Solution about an Ordinary Point

15.1 Example

Find a series solution around $x = x_0 = 0$ to the differential equation $(1-x)y'' + xy' - y = 0$.

The coefficients in the differential equation are:

$$\begin{aligned} P(x) &= 1 - x \\ Q(x) &= x \\ R(x) &= -1 \end{aligned}$$

These are already in powers of $(x-0)$, so we do not need to do any Taylor series expansions of the coefficients.

Since $p(x) = x/(1-x)$ and $q(x) = -1/(1-x)$ are analytic at $x = 0$, $x_0 = 0$ is an ordinary point.

Assume a solution is of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

We assume the series will converge for some ρ, $|x| < \rho$. We will find ρ later.

Differentiate:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substitute into the differential equation:

$$(1-x) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

First, get the same power of x in each term. Replace $m = n+1$ in the two middle terms:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{m=1}^{\infty} (m+1) m a_{m+1} x^m + \sum_{m=1}^{\infty} m a_m x^m - \sum_{n=0}^{\infty} a_n x^n = 0$$

Relabel $m = n$:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + \sum_{n=1}^{\infty}na_nx^n - \sum_{n=0}^{\infty}a_nx^n = 0$$

Secondly, get all the summations starting at the same point. Generally, choose the highest and make all the summations start there. In this case we take out the $n = 0$ terms of the first and last terms:

$$2 \cdot 1\, a_2x^0 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + \sum_{n=1}^{\infty}na_nx^n - a_0x^0 - \sum_{n=1}^{\infty}a_nx^n = 0$$

Now collect all the terms together:

$$(2a_2 - a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + na_n - a_n]x^n = 0$$

We set the coefficients of x equal to zero (since the entire series equals zero). This is really equating powers of x, so keep that in mind if you are equating two series!

$$\begin{array}{l} 2a_2 - a_0 = 0, \quad n = 0 \\ n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + na_n - a_n = 0, \quad n = 1, 2, 3, \ldots \end{array}$$

Notice that if we take $n = 0$ in the second relation, we get $2a_2 - a_0 = 0$, so we can combine these two relations. This is <u>not</u> always going to happen! The <u>recurrence relation</u> is therefore:

$$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + na_n - a_n = 0, \quad n = 0, 1, 2, 3, \ldots$$

Now, use the recurrence relation to determine the coefficients a_n. $n = 0$ specifies a_2 in terms of a_1 and a_0. Hence, a_0 and a_1 are arbitrary. They represent the constants of integration.

$$\begin{aligned} a_{n+2} &= \frac{(n+1)n\, a_{n+1} - (n-1)a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, 3 \\ a_0 &= \text{arbitrary} \\ a_1 &= \text{arbitrary} \\ a_2 &= \frac{a_0}{2!} \\ a_3 &= \frac{2a_2}{3 \cdot 2} = \frac{2}{3 \cdot 2}\frac{a_0}{2!} = \frac{a_0}{3!} \\ a_4 &= \frac{3 \cdot 2\, a_3 - a_2}{4 \cdot 3} = \frac{3 \cdot 2\frac{a_0}{3!} - \frac{a_0}{2!}}{4 \cdot 3} = \frac{a_0}{4!} \\ a_5 &= \frac{4 \cdot 3\, a_4 - 2\, a_3}{5 \cdot 4} = \frac{4 \cdot 3\frac{a_0}{4!} - 2\frac{a_0}{3!}}{5 \cdot 4} = \frac{a_0}{5!} \end{aligned}$$

In general, we have

$$a_n = \frac{a_0}{n!}, \quad n = 0, 2, 3, 4, \ldots$$

Note that the $n = 1$ term is not included in the above. The solution to the differential equation is:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} \frac{a_0}{n!} x^n \\ &= a_0 \left[1 + \sum_{n=2}^{\infty} \frac{a_0}{n!} x^n \right] + a_1 x \\ &= a_0 y_1(x) + a_1 y_2(x) \end{aligned}$$

As we mentioned before, the constants of integration are a_0, a_1, which we previously called c_1, c_2. A fundamental set of solutions is $\{y_1, y_2\}$,

$$y_1 = 1 + \sum_{n=2}^{\infty} \frac{a_0}{n!} x^n, \quad y_2 = x.$$

The solution y_2 is obviously converged for all x. The convergence of y_1 can be found using the ratio test.

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 < 1 \text{ for all } x.$$

The radius of convergence is therefore $\rho = \infty$.

16 Series Solution about a Regular Singular Point

The solution depends on the roots of the indicial equation.

Consider $P(x)y'' + Q(x)y' + R(x)y = 0$ in the neighbourhood of the regular singular point $x_0 = 0$. Write the equation in the following form:

$$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0.$$

We have the associated Euler equation:

$$x^2 y'' + x p_0 y' + q_0 y = 0.$$

We seek a solution of the form:

$$y = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad a_0 \neq 0$$

Differentiate the series and substitute into the differential equation:

$$L[\phi](r,x) = a_0 F(r) x^r + \sum_{n=1}^{\infty} \left\{ F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] \right\} x^{r+n} = 0$$

where

$$F(r) = r(r-1) + p_0 r + q_0.$$

The coefficients in each power of x must be zero! The x^r term gives us the indicial equation:

$$F(r) = r(r-1) + p_0 r + q_0 = 0.$$

This is where the $a_0 \neq 0$ requirement becomes important!

This is exactly the indicial equation we would have found from the associated Euler equation. The roots of the indicial equation r_1, r_2 are called the **exponents of the singularity**. They determine the qualitative behaviour of the solution in the neighbourhood of the singularity.

These are easy to find, since we must simply solve the quadratic indicial equation for the associated Euler equation:

$$r(r-1) + p_0 r + q_0 = 0,$$

where we obtain p_0 and q_0 from the expansions we must obtain anyway to solve the problem.

Setting the x^{r+n} term equal to zero gives us the recurrence relation:

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0, \qquad n \geq 1$$

Note: a_n depends on r and **all** the preceding coefficients!

Note: The a_n can be calculated provided $F(r+n) \neq 0$.

16.1 If r_1, r_2 are complex

We will have no problems with division by zero, and we will obtain two complex valued solutions for which $r_{1,2} = \lambda \pm \mu i$. We can get real valued solutions by taking the real and imaginary parts of the solution as we have done before.

16.2 Real Roots

The roots of the indicial equation are $r_{1,2}$; assume that $r_1 \geq r_2$.
Therefore $r_1 + n \neq r_1$ or $r_1 + n \neq r_2$ for $n \geq 1$.
So $F(r_1 + n) \neq 0$ for $n \geq 1$, and we can always determine **one** solution of the form:

$$y_1(x) = x^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \qquad x > 0.$$

We have labeled the $a_n(r_1)$ to indicate the dependence of the recurrence relation (and hence the a_n) on the root of the indicial equation. We have taken the constant $a_0 = 1$.

16.3 If $r_2 \neq r_1$ and $r_1 - r_2$ is not a positive integer

Then $r_2 + n \neq r_1$ and $r_2 + n$ is never a root of the indicial equation ($F(r_2 + n) \neq 0$), and we get a second solution:

$$y_2(x) = x^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right], \qquad x > 0.$$

16.4 Repeated Roots $r_1 = r_2$

In this case, we can write the second solution as

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} b_n x^n.$$

Substitute into the differential equation in the usual way to obtain the b_n.

Another way of proceeding is to use reduction of order. The second solution is then given by

$$y_2(x) = y_1(x)\int \frac{\exp(-\int p(x)dx)}{y_1^2(x)}dx.$$

16.5 Roots that differ by an integer: $r_1 - r_2 = N$

In this case, we can write the second solution as

$$y_2(x) = cy_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} c_n x^n.$$

Substitute into the differential equation in the usual way to obtain the c_n. The constant c could be zero.

Another way of proceeding is to use reduction of order. The second solution is then given by

$$y_2(x) = y_1(x) \int \frac{\exp(-\int p(x)dx)}{y_1^2(x)} dx.$$

Convergence: Consider the series alone, with out the $x^{r_{1,2}}$ part. As before, these series will converge at least with the radius of convergence of the minimum of the $xp(x)$ and $x^2q(x)$ radius of convergence. These functions are analytic at $x_0 = 0$. The singular behaviour, if any, is entirely contained in the $x^{r_{1,2}}$ factor!

To go to negative x, we end up with the same equations, so we can replace $x \to |x|$ and consider all $x \neq 0$.

16.6 Example

Solve the differential equation $3xy'' + y' - y = 0$ with a series solution about $x = 0$.

Identify

$$p(x) = \frac{1}{3x}, \qquad q(x) = -\frac{1}{3x}$$

Therefore, $x = 0$ is a singular point.

$$xp(x) = \frac{1}{3}, \qquad x^2q(x) = -\frac{x}{3}$$

which are analytic about $x = 0$, so $x = 0$ is a regular singular point.

Assume a solution looks like

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

Aside: Compare what is going on with what happens for the series solution about an ordinary point:

Ordinary	Regular Singular
$y = a_0 + a_1x + a_2x^2 + \dots$	$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \dots$
$y' = a_1 + 2a_2x + \dots$	$y' = ra_0x^{r-1} + (r+1)a_1x^r + (r+2)a_2x^{r+1} + \dots$

Notice that the a_0 is still involved in the derivative! This is because each terms in y has an x^r in it.

Substitute into the differential equation:
$P(x) = 3x$, $Q(x) = 1$, and $R(x) = -1$ are already in powers of x^n.

$$
\begin{aligned}
&3x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + \sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} - \sum_{n=0}^{\infty}a_nx^{n+r} = 0\\
&\sum_{n=0}^{\infty}(n+r)(3n+3r-3)a_nx^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1} - \sum_{n=0}^{\infty}a_nx^{n+r} = 0\\
&\sum_{n=0}^{\infty}(n+r)(3n+3r-2)a_nx^{n+r-1} - \sum_{n=0}^{\infty}a_nx^{n+r} = 0\\
&x^r[\sum_{n=0}^{\infty}(n+r)(3n+3r-2)a_nx^{n-1} - \sum_{n=0}^{\infty}a_nx^{n}] = 0\\
&x^r[r(3r-2)a_0x^{-1} + \sum_{n=1}^{\infty}(n+r)(3n+3r-2)a_nx^{n-1} - \sum_{n=0}^{\infty}a_nx^{n}] = 0\\
&x^r[r(3r-2)a_0x^{-1} + \sum_{n=0}^{\infty}(n+r+1)(3n+3r+1)a_{n+1}x^{n} - \sum_{n=0}^{\infty}a_nx^{n}] = 0\\
&x^r[r(3r-2)a_0x^{-1} + \sum_{n=0}^{\infty}\left\{(n+r+1)(3n+3r+1)a_{n+1} - a_n\right\}x^{n}] = 0
\end{aligned}
$$

We can now identify the indicial equation and the recurrence relation:
Indicial equation: $r(3r-2) = 0$, and $a_0 \neq 0$.
Recurrence equation: $(n+r+1)(3n+3r+1)a_{n+1} - a_n = 0$, $n = 0, 1, 2, \ldots$.

Note: if $a_0 = 0$, we do not get an indicial equation since the indicial equation is automatically satisfied for all r. This is precisely where the restriction $a_0 \neq 0$ arises from.

The indicial equation has two roots, $r_1 = 2/3$ and $r_2 = 0$. We want to work out a series solutions for each of the roots of the indicial equation. These are two distinct roots *not differing by an integer*, so we will get two solutions.

$r_1 = 2/3$:
The recurrence relation becomes:

$$
\begin{aligned}
a_{n+1} &= \frac{a_n}{(3n+5)(n+1)}, n = 0, 1, 2, 3, \ldots\\
a_0 &= \text{ arbitrary}\\
a_1 &= \frac{a_0}{5\cdot 1}\\
a_2 &= \frac{a_1}{8\cdot 2} = \frac{a_0}{2!\,5\cdot 8}\\
a_3 &= \frac{a_2}{11\cdot 3} = \frac{a_0}{3!\,5\cdot 8\cdot 11}\\
a_4 &= \frac{a_3}{14\cdot 4} = \frac{a_0}{4!\,5\cdot 8\cdot 11\cdot 14}\\
a_n &= \frac{a_0}{n!\,5\cdot 8\cdot 11\cdot 14\cdots(3n+2)}; \quad n = 1, 2, 3, \ldots
\end{aligned}
$$

$$
y_1 = a_0x^{2/3}\left[1 + \sum_{n=1}^{\infty}\frac{x^n}{n!\,5\cdot 8\cdot 11\cdot 14\cdots(3n+2)}\right]
$$

But, a_0 is arbitrary, so let's set $a_0 = 1$. This is the first of two solutions which will form a fundamental set of solutions.

$\underline{r_1 = 0}$:
The recurrence relation becomes:

$$\begin{aligned} a_{n+1} &= \frac{a_n}{(n+1)(3n+1)}, n = 0, 1, 2, 3, \ldots \\ a_0 &= \text{ arbitrary} \\ a_1 &= \frac{a_0}{1 \cdot 1} \\ a_2 &= \frac{a_1}{2 \cdot 4} = \frac{a_0}{2!\, 1 \cdot 4} \\ a_3 &= \frac{a_2}{3 \cdot 7} = \frac{a_0}{3!\, 1 \cdot 4 \cdot 7} \\ a_4 &= \frac{a_3}{4 \cdot 10} = \frac{a_0}{4!\, 1 \cdot 4 \cdot 7 \cdot 10} \\ a_n &= \frac{a_0}{n!\, 1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}; \quad n = 1, 2, 3, \ldots \end{aligned}$$

$$y_2 = a_0 x^0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\, 1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)} \right]$$

But, a_0 is arbitrary, so let's set $a_0 = 1$. This is the second of two solutions which will form a fundamental set of solutions.

Note: Here the recurrence relations did not generate a series solution that split into two solutions based on factoring out an a_0 and an a_1. We got our two solutions via the two roots of the indicial equation. This is a fundamental difference over how we got the general solution for a series solution about an ordinary point.

The y_1 and y_2 are linearly independent, since the powers of x are different. The ratio test will show the series converge for all x. Alternately, you could look at the complex poles of $p(x)$ and $q(x)$, note that the only pole is at $x = 0$, and therefore the minimum radius of convergence of our solutions must be $\rho = \infty$.

The general solution is therefore:

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ y_1 &= x^{2/3} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\, 5 \cdot 8 \cdot 11 \cdot 14 \cdots (3n+2)} \right] \\ y_2 &= \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\, 1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)} \right] \end{aligned}$$

17 Systems of Differential Equations

17.1 Basic Theory

Consider the system of linear 1st order equations: $\mathbf{x}' = \mathbf{p}(t)\mathbf{x} + \mathbf{g}(t)$, where: $\mathbf{x}$: n vector, $\mathbf{g}(t)$: n vector, $\mathbf{p}(t)$: $n \times n$ matrix. If $\mathbf{g}(t) = \mathbf{0}$ the system is homogeneous. If $\mathbf{p}(t) = \mathbf{A}$ constant, we have constant coefficients.

An initial value problem contains the initial conditions: $\mathbf{x}(t_0) = \mathbf{x}_0$.

Homogeneous: $\mathbf{x}' = \mathbf{p}(t)\mathbf{x}$

The solutions are written as

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, \ldots, \mathbf{x}^{(k)} = \begin{pmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{pmatrix}, \ldots, \mathbf{x}^{(n)} = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}.$$

so x_{ij} refers to the i^{th} component of the j^{th} solution $\mathbf{x}^{(j)}$.

The Superposition Principle. If $\mathbf{x}^{(k_1)}$ and $\mathbf{x}^{(k_2)}$ are solutions to a system of linear differential equations, then $c_1\mathbf{x}^{(k_1)} + c_2\mathbf{x}^{(k_2)}$ is also a solution for constant c_1 and c_2.

Linear Independence and The Wronskian

If we have the n solution vectors $\mathbf{x}^{(i)}$, $i = 1, \ldots, n$, then we define the fundamental matrix:

$$\Psi(t) = \mathbf{X}(t) = \left(\mathbf{x}^{(1)} \cdots \mathbf{x}^{(n)}\right) = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

The Wronskian is defined as the determinant of the fundamental matrix, $W(t) = \det \mathbf{X}$. If $W(t) \neq 0$, then the $\mathbf{x}^{(i)}$ are linearly independent.

A fundamental set of solutions is $\mathbf{x}^{(i)}$, $i = 1, \ldots, n$, when the $\mathbf{x}^{(i)}$ are linearly independent and solutions of the system of differential equations. A general solution is constructed from a fundamental set as $\mathbf{x} = \sum_{i=1}^{n} c_i\mathbf{x}^{(i)}$, for c_i constant.

Nonhomogeneous: $\mathbf{x}' = \mathbf{p}(t)\mathbf{x} + \mathbf{g}(t)$

The general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$ where:
$\mathbf{x}_c$, the complementary solution, is the general solution of the associated homogeneous problem $\mathbf{x}' = \mathbf{p}(t)\mathbf{x}$.
$\mathbf{x}_p$ is any particular solution, which we can find by

- diagonalization,
- undetermined coefficients,
- variation of parameters.

18 Homogeneous Systems with Constant Coefficients

18.1 Example: Real Eigenvalues

Solve the initial value problem:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \mathbf{x}, \;\; \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Assume that $\mathbf{x} = \xi e^{\lambda t}$, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{array}{rl} & \mathbf{x} = \xi e^{\lambda t} \\ & \mathbf{x}' = \lambda \xi e^{\lambda t} \\ \text{Substitute:} & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\ e^{\lambda t} \neq 0: & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \text{ or } \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}$$

Eigenvalues:

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= 0 \\ (5-\lambda)(1-\lambda+3 &= 0 \\ 5 - \lambda - 5\lambda + \lambda^2 + 3 &= 0 \\ \lambda^2 - 6\lambda + 8 &= 0 \\ (\lambda - 4)(\lambda - 2) &= 0 \end{aligned}$$

The eigenvalues are $\lambda^{(1)} = 4$ and $\lambda^{(2)} = 2$.

Eigenvectors:

$\lambda^{(1)} = 4$:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)} I)\xi^{(1)} &= \mathbf{0} \\ (\mathbf{A} - 4I)\xi^{(1)} &= \mathbf{0} \\ \begin{pmatrix} 5-4 & -1 \\ 3 & 1-4 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} \xi_1^{(1)} - \xi_2^{(1)} &= 0 \\ 3\xi_1^{(1)} - 3\xi_2^{(1)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(1)}$ to be arbitrary. Set $\xi_2^{(1)} = 1$.
Therefore, $\xi_1^{(1)} = \xi_2^{(1)} = 1$.

$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = 4$.

A solution of the system of differential equations is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$.

<u>$\lambda^{(2)} = 2$</u>:

$$\begin{aligned} (\mathbf{A} - \lambda^{(2)} I)\xi^{(2)} &= \mathbf{0} \\ (\mathbf{A} - 2I)\xi^{(2)} &= \mathbf{0} \\ \begin{pmatrix} 5-2 & -1 \\ 3 & 1-2 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} 3\xi_1^{(2)} - \xi_2^{(2)} &= 0 \\ 3\xi_1^{(2)} - \xi_2^{(2)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(2)}$ to be arbitrary. Set $\xi_2^{(2)} = 3$.
Therefore, $\xi_1^{(2)} = \frac{1}{3}\xi_2^{(2)} = 1$.

$\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(2)} = 2$.

A solution of the system of differential equations is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$.

<u>The General Solution</u>:
Check linear independence by computing the Wronskian:

$$W(t) = \det \Psi(t) = \det \left(\mathbf{x}^{(1)} \mathbf{x}^{(2)} \right) = \begin{vmatrix} e^{4t} & e^{2t} \\ e^{4t} & 3e^{2t} \end{vmatrix} = 3e^{6t} - e^{6t} = 2e^{6t} \neq 0$$

Therefore, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent. Therefore, they form a fundamental set of solutions.

A general solution is therefore

$$\begin{aligned} \mathbf{x} &= c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} \\ &= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} \end{aligned}$$

The general solution could also be written in terms of the fundamental matrix and a constant vector **c**. This constant vector represents the constants of integration.

$$\mathbf{x} = \begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & 3e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Psi(t)\mathbf{c}$$

<u>Solve the initial value problem</u>:

Apply the initial condition and determine the constant vector **c**:

$$\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \Psi(0)\mathbf{c} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Use Cramer's Rule to solve for c_1 and c_2:

$$c_1 = \frac{\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}} = \frac{6+1}{3-1} = \frac{7}{2}$$

$$c_2 = \frac{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}{2} = \frac{-1-2}{2} = -\frac{3}{2}$$

The solution to the initial value problem is therefore

$$\mathbf{x} = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} = \begin{pmatrix} e^{4t} & e^{2t} \\ e^{4t} & 3e^{2t} \end{pmatrix} \begin{pmatrix} \frac{7}{2} \\ -\frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{2}e^{4t} - \frac{3}{2}e^{2t} \\ \frac{7}{2}e^{4t} - \frac{9}{2}e^{2t} \end{pmatrix}$$

18.2 Example: Complex Eigenvalues

Solve the system of differential equations:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}.$$

Assume that $\mathbf{x} = \xi e^{\lambda t}$, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{aligned} & \mathbf{x} = \xi e^{\lambda t} \\ & \mathbf{x}' = \lambda \xi e^{\lambda t} \\ \text{Substitute:}\quad & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\ e^{\lambda t} \neq 0\text{:}\quad & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \text{ or } \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Eigenvalues:

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= 0 \\ \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} &= 0 \\ (3-\lambda)(-1-\lambda)+8 &= 0 \\ \lambda^2 - 2\lambda + 5 &= 0 \\ (\lambda - (1+2i))(\lambda - (1-2i)) &= 0 \end{aligned}$$

The eigenvalues are $\lambda^{(1)} = 1+2i$ and $\lambda^{(2)} = 1-2i$. The eigenvalues are complex conjugates since the matrix $\mathbf{A}$ is real valued.

Eigenvectors:

$\lambda^{(1)} = 1+2i$:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)} I)\xi^{(1)} &= \mathbf{0} \\ (\mathbf{A} - (1+2i)I)\xi^{(1)} &= \mathbf{0} \\ \begin{pmatrix} 3-1-2i & -2 \\ 4 & 1-1-2i \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} (2-2i)\xi_1^{(1)} - 2\xi_2^{(1)} &= 0 \\ 4\xi_1^{(1)} - (2+2i)\xi_2^{(1)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_1^{(1)}$ to be arbitrary. Set $\xi_1^{(1)} = 1$.
Therefore, $\xi_2^{(1)} = (1-i)\xi_1^{(1)} = 1-i$.

$\xi^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = 1+2i$.

The eigenvalues and eigenvectors occur in complex conjugate pairs for a real valued matrix, so we can immediately say:

$\xi^{(2)} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(2)} = 1-2i$.

Complex valued solutions of the system of differential equations are
$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t}$, and
$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(1-2i)t}$.

To get real valued solutions, we can split $\mathbf{x}^{(1)}$ into real and complex parts. Each of these will be a solution, and each will be real valued.

$$\begin{aligned}
\mathbf{x}^{(1)} &= \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t} \\
&= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} i \right] [\cos 2t + i \sin 2t]\, e^t \\
&= \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \sin 2t \right] + i \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \sin 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \cos 2t \right] \\
&= \mathbf{u}(t) + i\mathbf{v}(t)
\end{aligned}$$

Two real valued solutions are $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

$$\begin{aligned}
\mathbf{u}(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \cos 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \sin 2t \\
\mathbf{v}(t) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \sin 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \cos 2t
\end{aligned}$$

<u>The General Solution</u>:
Check linear independence by computing the Wronskian:

$$W(t) = \det \Psi(t) = \det (\mathbf{u} \;\; \mathbf{v}) = \begin{vmatrix} e^t \cos 2t & e^t \sin 2t \\ e^t \cos 2t + e^t \sin 2t & e^t \sin 2t - e^t \cos 2t \end{vmatrix} = -e^{2t} \neq 0$$

Therefore, $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Therefore, they form a fundamental set of solutions. A general solution is therefore $\mathbf{x} = c_1\mathbf{u}(t) + c_2\mathbf{v}(t)$.

18.3 Example: Repeated Eigenvalues

Solve the system of differential equations:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{x}.$$

Assume that $\mathbf{x} = \xi e^{\lambda t}$, where ξ is a constant 3-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{aligned}
& \mathbf{x} = \xi e^{\lambda t} \\
& \mathbf{x}' = \lambda \xi e^{\lambda t} \\
\text{Substitute:} \quad & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\
e^{\lambda t} \neq 0: \quad & (\mathbf{A} - \lambda I)\xi = \mathbf{0}
\end{aligned}$$

<u>Eigenvalues</u>:

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= 0 \\ \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} &= 0 \\ -(\lambda+1)^2(\lambda-5) &= 0 \end{aligned}$$

The eigenvalues are $\lambda^{(1)} = -1$, $\lambda^{(2)} = -1$, and $\lambda^{(3)} = 5$. We have an eigenvalue of multiplicity two.

<u>Eigenvectors</u>:

<u>$\lambda^{(1)} = \lambda^{(2)} = -1$</u>:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)} I)\xi &= \mathbf{0} \\ (\mathbf{A} - I)\xi &= \mathbf{0} \\ \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the three equations:

$$\begin{aligned} 2\xi_1 - 2\xi_2 + 2\xi_3 &= 0 \\ -2\xi_1 + 2\xi_2 - 2\xi_3 &= 0 \\ 2\xi_1 - 2\xi_2 + 2\xi_3 &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 3 unknowns.
Choose ξ_2 <u>and</u> ξ_3 to be arbitrary.
Therefore, $\xi_1 = \xi_2 - \xi_3$.

Set $\xi_2 = 1, \xi_3 = 0$. Therefore, $\xi_1 = 1$.

$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = -1$.

Set $\xi_2 = 0, \xi_3 = 1$. Therefore, $\xi_1 = -1$.

$\xi^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(2)} = -1$.

We were able to obtain two linearly independent (they are not multiples of each other) eigenvectors even though we only had a single eigenvalue! Two linearly independent solutions are
$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t}$, and
$\mathbf{x}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}$.

To get the third solution, use the third eigenvalue.

$\underline{\lambda^{(3)} = 5}$:

$$\begin{aligned}(\mathbf{A} - \lambda^{(3)} I)\xi &= \mathbf{0}\\ (\mathbf{A} - 5I)\xi &= \mathbf{0}\\ \begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

So we get the three equations:

$$\begin{aligned}-4\xi_1 - 2\xi_2 + 2\xi_3 &= 0\\ -2\xi_1 - 4\xi_2 - 2\xi_3 &= 0\\ 2\xi_1 - 2\xi_2 - 4\xi_3 &= 0\end{aligned}$$

You can use Gauss Jordan elimination to determine the solution to this system. Or, you could use the command Solve in *Mathematica*. The solution is

$$\begin{aligned}\xi_1 &= \xi_3\\ \xi_2 &= -\xi_3\\ \xi_3 &= \text{arbitrary}\end{aligned}$$

If we pick $\xi_3 = 1$, we have $\xi_1 = 1, \xi_2 = -1, \xi_3 = 1$.

$\xi^{(3)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(3)} = 5$.

A third solution to the system of differential equations is $\mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}$.

The general solution is therefore (you can check linear independence):
$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}$

18.4 Example: Repeated Eigenvalues with One Eigenvector

Solve the system of differential equations:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{x}.$$

Assume that $\mathbf{x} = \xi e^{\lambda t}$, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{array}{rl} & \mathbf{x} = \xi e^{\lambda t} \\ & \mathbf{x}' = \lambda \xi e^{\lambda t} \\ \text{Substitute:} & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\ e^{\lambda t} \neq 0\text{:} & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \end{array}$$

Eigenvalues:

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= 0 \\ \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} &= 0 \\ (\lambda+3)^2 &= 0 \end{aligned}$$

The eigenvalues are $\lambda^{(1)} = -3$, $\lambda^{(2)} = -3$. We have an eigenvalue of multiplicity two.

Eigenvectors:

$\lambda^{(1)} = \lambda^{(2)} = -3$:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)} I)\xi &= \mathbf{0} \\ (\mathbf{A} + 3I)\xi &= \mathbf{0} \\ \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} 6\xi_1 - 18\xi_2 &= 0 \\ 2\xi_1 - 6\xi_2 &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose ξ_2 to be arbitrary. Set $\xi_2 = 1$.
Therefore, $\xi_1 = 3\xi_2 = 3$.

$\xi^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = -3$.

One solution is
$\mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$.

We cannot get a second eigenvector for this eigenvalue like we did in the previous example. For a second order linear differential equation, if we had a repeated root of the characteristic equation we could show a second solution existed which looked like te^{rt}, in other words, t times the first solution. We want to do something similar here, but we need to modify things just a bit. For systems, we assume a second solution exists which looks like:

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)}t + \eta e^{\lambda^{(1)}t}.$$

Notice that the only difference from what we did before was that we now add on a extra term that looks like the first solution, where η is a 2-vector constant. Now we must determine η.

If you substitute this into the differential equation you will find:

$$(\mathbf{A} - \lambda^{(1)}I)\xi^{(1)}e^{\lambda^{(1)}t}t + (\mathbf{A}\eta - \eta\lambda^{(1)} - \xi^{(1)})e^{\lambda^{(1)}t} = 0$$

Setting the coefficients of powers of t to zero, and noting that $e^{\lambda^{(1)}t} \neq 0$, gives us the two equations:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)}I)\xi^{(1)} &= 0 \\ (\mathbf{A} - \lambda^{(1)}I)\eta &= \xi^{(1)} \end{aligned}$$

The first equation we have already solved. We must solve the second equation for η.

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)}I)\eta &= \xi^{(1)} \\ (\mathbf{A} + 3I)\eta &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} 6\eta_1 - 18\eta_2 &= 3 \\ 2\eta_1 - 6\eta_2 &= 1 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose η_2 to be arbitrary. Set $\eta_2 = 0$.
Therefore, $\eta_1 = \frac{1}{2}$.

A second solution is therefore

$$\begin{aligned} \mathbf{x}^{(2)} &= \mathbf{x}^{(1)}t + \eta e^{\lambda^{(1)}t} \\ &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{-3t} \end{aligned}$$

The general solution is $\mathbf{x} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$.

19 Nonhomogeneous Systems with Constant Coefficients: Undetermined Coefficients

19.1 Example

Solve the linear differential system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

First, get the complimentary solution by solving the associated homogeneous equation:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$$

Assume that $\mathbf{x} = \xi e^{\lambda t}$, where ξ is a constant 2-vector and λ is a constant scalar. Differentiate and substitute into the differential system:

$$\begin{array}{rl} & \mathbf{x} = \xi e^{\lambda t} \\ & \mathbf{x}' = \lambda \xi e^{\lambda t} \\ \text{Substitute:} & \lambda \xi e^{\lambda t} = \mathbf{A} \xi e^{\lambda t} \\ e^{\lambda t} \neq 0\text{:} & (\mathbf{A} - \lambda I)\xi = \mathbf{0} \\ & \begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}$$

<u>Eigenvalues</u>:

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= 0 \\ \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} &= 0 \\ (2-\lambda)(-2-\lambda) + 3 &= 0 \\ \lambda^2 - 1 &= 0 \\ (\lambda+1)(\lambda-1) &= 0 \end{aligned}$$

The eigenvalues are $\lambda^{(1)} = -1$ and $\lambda^{(2)} = +1$.

<u>Eigenvectors</u>:

<u>$\lambda^{(1)} = -1$</u>:

$$\begin{aligned} (\mathbf{A} - \lambda^{(1)} I)\xi^{(1)} &= \mathbf{0} \\ (\mathbf{A} + I)\xi^{(1)} &= \mathbf{0} \\ \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} 3\xi_1^{(1)} - \xi_2^{(1)} &= 0 \\ 3\xi_1^{(1)} - \xi_2^{(1)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(1)}$ to be arbitrary. Set $\xi_2^{(1)} = 3$.
Therefore, $\xi_1^{(1)} = \frac{1}{3}\xi_2^{(1)} = 1$.

$\xi^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(1)} = -1$.

A solution of the system of differential equations is $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$.

$\lambda^{(2)} = +1$:

$$\begin{aligned} (\mathbf{A} - \lambda^{(2)} I)\xi^{(2)} &= \mathbf{0} \\ (\mathbf{A} - I)\xi^{(2)} &= \mathbf{0} \\ \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So we get the two equations:

$$\begin{aligned} \xi_1^{(2)} - \xi_2^{(2)} &= 0 \\ 2\xi_1^{(2)} - 2\xi_2^{(2)} &= 0 \end{aligned}$$

These are the same equation. So we have 1 equation with 2 unknowns.
Choose $\xi_2^{(2)}$ to be arbitrary. Set $\xi_2^{(2)} = 1$.
Therefore, $\xi_1^{(2)} = \xi_2^{(2)} = 1$.

$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector associated with the eigenvalue $\lambda^{(2)} = +1$.

A solution of the system of differential equations is $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}$.

The Complementary Solution:

$$W(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})(t) = \det\left(\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t)\right) = \begin{vmatrix} e^{-t} & e^{3t} \\ 3e^{-t} & e^{3t} \end{vmatrix} = e^{2t} - 3e^{2t} = -2e^{2t} \neq 0$$

Since the Wronskian is not zero, the two solutions we have found are linearly independent and form a fundamental set of solutions.

A complementary solution is therefore

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} \\ &= c_1\begin{pmatrix}1\\3\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{t}\end{aligned}$$

Use undetermined coefficients to find a particular solution. First, rewrite the nonhomogeneous term $\mathbf{g}(t)$:

$$\mathbf{g}(t) = \begin{pmatrix}e^t\\t\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}t + \begin{pmatrix}1\\0\end{pmatrix}e^t$$

We do this so we can see what type of solution we should assume exists. Since this is a polynomial added to an exponential, we assume a solution looks like:

$$\mathbf{x} = \mathbf{v} = \mathbf{a}t + \mathbf{b} + \mathbf{c}e^t$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are 2 vectors. Is any part of this included in any part of the complementary solution? Yes–the $\mathbf{c}e^t$ appears in the complementary solution. Therefore, we know this assumed form of the solution will not work.

Instead, let's assume:

$$\mathbf{x} = \mathbf{v} = \mathbf{a}t + \mathbf{b} + \mathbf{c}te^t + \mathbf{d}e^t$$

Notice the inclusion of the last term. This is different from what we would have assumed before. We need to find four equations in the four unknowns $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

Differentiate and substitute into the system of differential equations:

$$\begin{aligned}\mathbf{v} &= \mathbf{a}t + \mathbf{b} + \mathbf{c}te^t + \mathbf{d}e^t \\ \mathbf{v}' &= \mathbf{a} + \mathbf{c}e^t + \mathbf{c}te^t + \mathbf{d}e^t \\ \mathbf{a} + \mathbf{c}e^t + \mathbf{c}te^t + \mathbf{d}e^t &= \mathbf{A}(\mathbf{a}t + \mathbf{b} + \mathbf{c}te^t + \mathbf{d}e^t) + \begin{pmatrix}0\\1\end{pmatrix}t + \begin{pmatrix}1\\0\end{pmatrix}e^t\end{aligned}$$

Collect terms:

$$(\mathbf{Ab} - \mathbf{a}) + t\left(\mathbf{Aa} + \begin{pmatrix}0\\1\end{pmatrix}\right) + e^t\left(\mathbf{Ad} + \begin{pmatrix}1\\0\end{pmatrix} - \mathbf{d} - \mathbf{c}\right) + te^t\left(\mathbf{Ac} - \mathbf{c}\right) = \mathbf{0}$$

And we get the four equations which must be satisfied:

$$\begin{aligned} \mathbf{Ab} - \mathbf{a} &= \mathbf{0} \quad (1) \\ \mathbf{Aa} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \mathbf{0} \quad (2) \\ \mathbf{Ad} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{d} - \mathbf{c} &= \mathbf{0} \quad (3) \\ \mathbf{Ac} - \mathbf{c} &= \mathbf{0} \quad (4) \end{aligned}$$

Equation (2):

$$\mathbf{Aa} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{0} \longrightarrow \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Use Cramer's Rule:

$$a_1 = \frac{\begin{vmatrix} 0 & -1 \\ -1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix}} = \frac{-1}{-4+3} = 1, \quad a_2 = \frac{\begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}}{-1} = \frac{-2}{-1} = 2, \longrightarrow \mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Equation (1):

$$\mathbf{Ab} - \mathbf{a} = \mathbf{0} \longrightarrow \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Use Cramer's Rule:

$$b_1 = \frac{\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix}} = \frac{-2+2}{-1} = 0, \quad b_2 = \frac{\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}}{-1} = \frac{4-3}{-1} = -1, \longrightarrow \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Equation (4):

$$\mathbf{Ac} - \mathbf{c} = \mathbf{0} \longrightarrow \mathbf{Ac} = \mathbf{c}$$

This means that $\mathbf{c}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $+1$. We already worked out the eigenvectors for $\mathbf{A}$ when we solved the associated homogeneous system!

$$\lambda^{(2)} = +1, \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \mathbf{c} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for any nonzero constant k.

Equation (3):

$$\mathbf{Ad} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{d} - \mathbf{c} = \mathbf{0}$$

$$(\mathbf{A} - I)\mathbf{d} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} k-1 \\ k \end{pmatrix}$$

Use Cramer's Rule:

$$d_1 = \frac{\begin{vmatrix} k-1 & -1 \\ k & -3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 3 & -3 \end{vmatrix}} = \frac{-3(k-1)+k}{0}$$

Division by zero! However, as long as the numerator is zero as well, we will be alright. We will have to pick a specific value for k. $k = 3/2$ makes the numerator zero, fixes $\mathbf{c} = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and we now have the system:

$$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}$$

We have the two equations:

$$\begin{aligned} d_1 - d_2 &= 1/2 \\ 3d_1 - 3d_2 &= 3/2 \end{aligned}$$

These are the same equations. We therefore have 1 equation in 2 unknowns.
Choose d_2 to be arbitrary. We could pick anything, but let's pick $d_2 = -3/4$. Then, $d_1 = 1/2 + d_2 = -1/4$.

$$\mathbf{d} = -\frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Therefore, a particular solution is

$$\mathbf{x}_p = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t - \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t$$

A general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$,

$$\begin{aligned} \mathbf{x} &= c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \mathbf{x}_p \\ &= c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t - \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t \end{aligned}$$