- The final exam will include two parts, Part A (10 marks) will be several True/False or Multiple Choice and Part B (140 marks) will be long answer in which you will choose 7 questions to answer from a list of 9.
- You will have 120 minutes to complete the final exam.
- Use the Study Guide, Assignments, Examples in Text, and WeBWorK to assist with your review.
- I will not be asking you to determine patterns in series solutions (it takes too long).

You will be provided with at least the following information on the final exam:

Laplace transform: $F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt.$

Step function: $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$ If f is periodic with period T, then $\mathcal{L}[f(t)] = \dfrac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$

Dirac Delta function: $\delta(t - t_0) = 0$ if $t \neq t_0$, $\int_{-\infty}^\infty \delta(t - t_0) f(t)\, dt = f(t_0)$

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$
9.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
10.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t) f(t-c)$	$e^{-cs} F(s)$
14.	$e^{ct} f(t)$	$F(s-c)$
15.	$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right), \quad c > 0$
17.	$\delta(t-c)$	e^{-cs}
18.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
18a.	$f'(t)$	$sF(s) - f(0)$
18b.	$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$
19.	$(-t)^n f(t)$	$F^{(n)}(s)$

Problem 1. Answer as True or False:

The function $f(x) = (1+x)^{-1}$ is analytic at $x_0 = -1$ .... T F

A series which converges for $|x - x_0| < \rho$ can be differentiated term by term,

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x-x_0)^n = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

<u>for all x</u> .... T F

The differential equation $y'' - 2xy = 0$ has two solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$ which converge for $|x| < \infty$
T F

The differential equation $2x^2y'' + 3xy' - y = 0$ has the solution $y = c_1 x^{1/2} + c_2 x^{-1}$ .... T F

The differential equation $(x+1)y'' - 2y = 0$ has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

which converge for $|x - 1| < 2$ .... T F

For $\delta(t)$ the the Dirac delta function, $\displaystyle\int_{-\infty}^{\infty} \delta(t - \pi/3)\sin t\, dt = \frac{\sqrt{3}}{2}$ .... T F

For $u_c(t)$ the Heaviside step function, $u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & -\infty < t < 5 \text{ and } t \ge 20 \end{cases}$ .... T F

Problem 2.

(a) Find a general solution of the differential equation $2x^2y'' + 3xy' - y = 0$, $x > 0$.

(b) Explain why the differential equation $(x+1)y'' - 2y = 0$ has

(i) two solutions of the form $y = \sum_{n=0}^{\infty} a_n(x-1)^n$,

(ii) the solutions will have a minimum radius of convergence of $\rho = 2$.

NOTE: you should not have to solve for the series to answer this question.

Problem 3. Determine the <u>recurrence relation</u> for the series solution about the ordinary point $x_0 = 1$ of the differential equation $y'' - xy = 0$. You do not have to find the series solutions, nor discuss convergence.

Problem 4. Determine the first four terms in the fundamental set of solutions for series solution to the DE about the regular singular point $x_0 = 0$.

$$2x^2y'' - xy' + (1+x)y = 0$$

Problem 5. Explain what is required for a function $f(x)$ to be analytic at x_0. With reference to the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

explain what ordinary, regular singular, and irregular singular points are.

Problem 6. Find the solution of the initial value problem

$$2x^2y'' + 3xy' - y = 0, \quad y(1) = 0, y'(1) = 1, x > 0.$$

Problem 7. Explain why the differential equation $(x^2 - x + 1)y'' + y = 0$ has two solutions of the form $y = \sum_{n=0}^{\infty} a_n(x+1)^n$ each with a minimum radius of convergence of $\rho = \sqrt{3}$.

NOTE: you should not have to solve for the series to answer this question.

Problem 8. Find the solution of the boundary value problem

$$x^2y'' + 3xy' - y = x^2, \quad y(1) = 0, y(e) = 0, x > 0.$$

Problem 9. Determine the Laplace transform of $f(t) = \sin at$ from the definition of the Laplace transform.

Problem 10. Using Laplace transforms, determine the solution to the initial value problem

$$y'' + 4y = \sin t + u_\pi(t)\sin(t - \pi), y(0) = 0, y'(0) = 0.$$

Hint: The partial fraction expansion you need will have the form $\dfrac{1}{(s^2+1)(s^2+4)} = \dfrac{A}{s^2+1} + \dfrac{B}{s^2+4}$.

Problem 11. For the 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, discuss (include sketches) the classification and stability of the point $\mathbf{x} = \mathbf{0}$ based on the eigenvalues of the matrix $\mathbf{A}$. Which situation models a physical system which has the property of conservation of energy?

Problem 12. Consider the nonlinear system:

$$x' = 2x + y^2,$$
$$y' = 3y - x.$$

Find the equilibrium point (x_0, y_0) that is not the origin. Linearize about (x_0, y_0) and then classify the behaviour of the solution around the point. Using the eigenvectors for the locally linear approximation, accurately sketch the behaviour of the solution to the nonlinear system around the equilibrium point. Make sure to include direction of motion in your sketch.

Problem 13. Consider the nonlinear system

$$\frac{dx}{dt} = x - xy$$
$$\frac{dy}{dt} = y + 2xy$$

- Determine the critical points of the system.
- Determine a solution of the form $H(x, y) = c$.
- Determine the value of c for the solution that passes through one of the equilibrium points you found.

Problem 14. Determine the first 4 nonzero terms in the fundamental set of solutions using series solution about $x = 0$ to the DE

$$y'' - xy = 0.$$

Problem 15. Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{A}$ is a 2×2 matrix. Below are four direction fields associated with four different matrices $\mathbf{A}$.

(a) Sketch a solution curve in the x_1x_2-plane for each case. Classify the point $\mathbf{x} = \mathbf{0}$ as either stable or unstable.

(b) Explain what you know about the eigenvalues for the matrix $\mathbf{A}$ in each case.